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(*b*. Basel, Switzerland, 15 April 1707; *d*. St. Petersburg, Russia, 18 September 1783)

*mathematics, mechanics, astronomy, physics*.

**Life** . Euler’s forebears settled in Basel at the end of the sixteenth century. His great-great-grandfather, Hans Georg Euler, had moved from Lindau, on the Bodensee (Lake Constance). They were, for the most part, artisans; but the mathematician’s father, Paul Euler, graduated from the theological department of the University of Basel. He became a Protestant minister, and in 1706 he married Margarete Brucker, daughter of another minister. In 1708 the family moved to the village of Riehen, near Basel, where Leonhard Euler spent his childhood.

Euler’s father was fond of mathematics and had attended Jakob Bernoulli’s lectures at the university; he gave his son his elementary education, including mathematics. In the brief autobiography dictated to his eldest son in 1767, Euler recollected that for several years he diligently and thoroughly studied Christoff Rudolf’s *Algebra*, a difficult work (dating, in Stifel’s edition, from 1553) which only a very gifted boy could have used. Euler later spent several years with his maternal grandmother in Basel, studying at a rather poor local Gymnasium; mathematics was not taught at all, so Euler studied privately with Johann Burckhardt, an amateur mathematician. In the autumn of 1720, being not yet fourteen, Euler entered the University of Basel in the department of arts to get a general education before specializing. The university was small; it comprised only a few more than a hundred students and nineteen professors. But among the latter was Johann I Bernoulli, who had followed his brother Jakob, late in 1705, in the chair of mathematics. During the academic year, Bernoulli delivered daily public lectures on elementary mathematics; besides that, for additional pay he conducted studies in higher mathematics and physics for those who were interested. Euler laboriously studied all the required subjects, but this did not satisfy him. According to the autobiography:

... I soon found an opportunity to be introduced to a famous professor Johann Bernoulli.... True, he was very busy and so refused flatly to give me private lessons; but he gave me much more valuable advice to start reading more difficult mathematical books on my own and to study them as diligently as I could; if I came across some obstacle or difficulty, I was given permission to visit him freely every Saturday afternoon and he kindly explained to me everything I could not understand.... and this, undoubtedly, is the best method to succeed in mathematical subjects.[1]

In the summer of 1722, Euler delivered a speech in praise of temperance, “De temperantia,” and received his *prima laurea*, a degree corresponding to the bachelor of arts. The same year he acted as opponent (*respondens*) at the defense of two theses—one on logic, the other on the history of law. In 1723 Euler received his master’s degree in philosophy. This was officially announced at a session on 8 June 1724; Euler made a speech comparing the philosophical ideas of Descartes and Newton. Some time earlier, in the autumn of 1723, he had joined the department of theology, fulfilling his father’s wish. His studies in theology, Greek, and Hebrew were not very successful, however; Euler devoted most of his time to mathematics. He finally gave up the idea of becoming a minister but remained a wholehearted believer throughout his life. He also retained the knowledge of the humanities that he acquired in the university; he had an outstanding memory and knew by heart the entirety of Vergil’s *Aeneid*. At seventy he could recall precisely the lines printed at the top and bottom of each page of the edition he had read when he was young.

At the age of eighteen, Euler began his independent investigations. His first work, a small note on the construction of isochronous curves in a resistant medium,[2] appeared in *Acta eruditorum* (1726); this was followed by an article in the same periodical on algebraic reciprocal trajectories (1727).[3] The problem of reciprocal trajectories was studied by Johann I Bernoulli, by his son Nikolaus II, and by other mathematicians of the time. Simultaneously Euler participated in a competition announced by the Paris Académie des Sciences which proposed for 1727 the problem of the most efficient arrangement of masts on a ship. The prize went to Pierre Bouguer, but Euler’s work[4] received the *accessit*. Later, from 1738 to 1772, Euler was to receive twelve prizes from the Academy.

For mathematicians beginning their careers in Switzerland, conditions were hard. There were few chairs of mathematics in the country and thus little chance of finding a suitable job. The income and public recognition accorded to a university professor of mathematics were not cause for envy. There were no scientific magazines, and publishers were reluctant to publish books on mathematics, which were considered financially risky. At this time the newly organized St. Petersburg Academy of Sciences (1725) was looking for personnel. In the autumn of that year Johann I Bernoulli’s sons, Nikolaus II and Daniel, went to Russia. On behalf of Euler, they persuaded the authorities of the new Academy to send an invitation to their young friend also.

Euler received the invitation to serve as adjunct of physiology in St. Petersburg in the autumn of 1726, and he began to study this discipline, with an effort toward applying the methods of mathematics and mechanics. He also attempted to find a job at the University of Basel. A vacancy occurred in Basel after the death of a professor of physics, and Euler presented as a qualification a small composition on acoustics, *Dissertatio physica de sono* (1727).[5] Vacancies were then filled in the university by drawing lots among the several chosen candidates. In spite of a recommendation from Johann Bernoulli, Euler was not chosen as a candidate, probably because he was too young—he was not yet twenty. But, as O. Spiess has pointed out, this was in Euler's favor;[6] a much broader field of action lay ahead of him.

On 5 April 1727 Euler left Basel for St. Petersburg, arriving there on 24 May. From this time his life and scientific work were closely connected with the St. Petersburg Academy and with Russia. He never returned to Switzerland, although he maintained his Swiss citizenship.

In spite of having been invited to St. Petersburg to study physiology, Euler was at once given the chance to work in his real field and was appointed an adjunct member of the Academy in the mathematics section. He became professor of physics in 1731 and succeeded Daniel Bernoulli, who returned to Basel in 1733 as a professor of mathematics. The young Academy was beset with numerous difficulties, but on the whole the atmosphere was exceptionally beneficial for the flowering of Euler's genius. Nowhere else could he have been surrounded by such a group of eminent scientists, including the analyst, geometer, and specialist in theoretical mechanics Jakob Hermann, a relative; Daniel Bernoulli, with whom Euler was connected not only by personal friendship but also by common interests in the field of applied mathematics; the versatile scholar Christian Goldbach, with whom Euler discussed numerous problems of analysis and the theory of numbers; F. Maier, working in trigonometry; and the astronomer and geographer J.-N. Delisle.

In St. Petersburg, Euler began his scientific activity at once. No later than August 1727 he started making reports on his investigations at sessions of the Academy; he began publishing them in the second volume of the academic proceedings, *Commentarii Academiae scientiarum imperialis Petropolitanae* (1727) (St. Petersburg, 1729). The generous publication program of the Academy was especially important for Euler, who was unusually prolific. In a letter written in 1749 Euler cited the importance that the work at the Academy had for many of its members:

... I and all others who had the good fortune to be for some time with the Russian Imperial Academy cannot but acknowledge that we owe everything which we are and possess to the favorable conditions which we had there.[7]

In addition to conducting purely scientific work, the St. Petersburg Academy from the very beginning was also obliged to educate and train Russian scientists, and with this aim a university and a Gymnasium were organized. The former existed for nearly fifty years and the latter until 1805. The Academy was also charged to carry out for the government a study of Russian territory and to find solutions for various technological problems. Euler was active in these projects. From 1733 on, he successfully worked with Delisle on maps in the department of geography. From the middle of the 1730's he studied problems of shipbuilding and navigation, which were especially important to the rise of Russia as a great sea power. He joined various technological committees and engaged in testing scales, fire pumps, saws, and so forth. He wrote articles for the popular periodical of the Academy and reviewed works submitted to it (including those on the quadrature of the circle), compiled the *Einleitung zur Rechen-Kunst*[8] for Gymnasiums, and also served on the examination board.

Euler's main efforts, however, were in the mathematical sciences. During his fourteen years in St. Petersburg he made brilliant discoveries in such areas as analysis, the theory of numbers, and mechanics. By 1741 he had prepared between eighty and ninety works for publication. He published fifty-five, including the two-volume *Mechanica*.[9]

As is usual with scientists, Euler formulated many of his principal ideas and creative concepts when he was young. Neither the dates of preparation of his works nor those of their actual publication adequately indicate Euler's intellectual progress, since a number of the plans formulated in the early years in St. Petersburg (and even as early as the Basel period) were not realized until much later. For example, the first drafts of the theory of motion of solid bodies, finished in the 1760's, were made during this time. Likewise Euler began studying hydromechanics while still in Basel, but the most important memoirs on the subject did not appear until the middle of the 1750's; he imagined a systematic exposition of differential calculus on the basis of calculus of finite differences in the 1730's but did not realize the intention until two decades later; and his first articles on optics appeared fifteen years after he began studying the subject in St. Petersburg. Only by a complete study of the unpublished Euler manuscripts would it be possible to establish the progression of his ideas more precisely.

Because of his large correspondence with scientists from many countries, Euler's discoveries often became known long before publication and rapidly brought him increasing fame. An index of this is Johann I Bernoulli's letters to his former disciple—in 1728 Bernoulli addressed the "most learned and gifted man of science Leonhard Euler"; in 1737 he wrote, the "most famous and wisest mathematician"; and in 1745 he called him the "incomparable Leonhard Euler" and *"mathematicorum princeps*." Euler was then a member of both the St. Petersburg and Berlin academies. (That certain frictions between Euler and Schumacher, the rude and despotic councillor of the St. Petersburg Academy, did Euler's career no lasting harm was due to his tact and diplomacy.) He was later elected a member of the Royal Society of London (1749) and the Académie des Sciences of Paris (1755). He was elected a member of the Society of Physics and Mathematics in Basel in 1753.

At the end of 1733 Euler married Katharina Gsell, a daughter of Georg Gsell, a Swiss who taught painting at the Gymnasium attached to the St. Petersburg Academy. Johann Albrecht, Euler's first son, was born in 1734, and Karl was born in 1740. It seemed that Euler had settled in St. Petersburg for good; his younger brother, Johann Heinrich, a painter, also worked there. His quiet life was interrupted only by a disease that caused the loss of sight in his right eye in 1738.

In November 1740 Anna Leopoldovna, mother of the infant Emperor Ivan VI, became regent, and the atmosphere in the Russian capital grew troubled. According to Euler's autobiography, "things looked rather dubious."[10] At that time Frederick the Great, who had succeeded to the Prussian throne in June 1740, decided to reorganize the Berlin Society of Sciences, which had been founded by Leibniz but allowed to degenerate during Frederick's father's reign. Euler was invited to work in Berlin. He accepted, and after fourteen years in Russia he sailed with his family on 19 June 1741 from St. Petersburg. He arrived in Berlin on 25 July.

Euler lived in Berlin for the next twenty-five years. In 1744 he moved into a house, still preserved, on the Behrenstrasse. The family increased with the birth of a third son, Christoph, and two daughters; eight other children died in infancy. In 1753 Euler bought an estate in Charlottenburg, which was then just outside the city. The estate was managed by his mother, who lived with Euler after 1750. He sold the property in 1763.

Euler's energy in middle age was inexhaustible. He was working simultaneously in two academies—Berlin and St. Petersburg. He was very active in transforming the old Society of Sciences into a large academy—officially founded in 1744 as the Académie Royale des Sciences et des Belles Lettres de Berlin. (The monarch preferred his favorite language, French, to both Latin and German.) Euler was appointed director of the mathematical class of the Academy and member of the board and of the committee directing the library and the publication of scientific works. He also substituted for the president, Maupertuis, when the latter was absent. When Maupertuis died in 1759, Euler continued to run the Academy, although without the title of president. Euler's friendship with Maupertuis enabled him to exercise great influence on all the activities of the Academy, particularly on the selection of members.

Euler's administrative duties were numerous: he supervised the observatory and the botanical gardens; selected the personnel; oversaw various financial matters; and, in particular, managed the publication of various calendars and geographical maps, the sale of which was a source of income for the Academy. The king also charged Euler with practical problems, such as the project in 1749 of correcting the level of the Finow Canal, which was built in 1744 to join the Havel and the Oder. At that time he also supervised the work on pumps and pipes of the hydraulic system at Sans Souci, the royal summer residence.

In 1749 and again in 1763 he advised on the organization of state lotteries and was a consultant to the government on problems of insurance, annuities, and widows' pensions. Some of Euler's studies on demography grew out of these problems. An inquiry from the king about the best work on artillery moved Euler to translate into German Benjamin Robins' *New Principles of Gunnery*. Euler added his own supplements on ballistics, which were five times longer than the original text (1745).[11] These supplements occupy an important place in the history of ballistics; Euler himself had written a short work on the subject as early as 1727 or 1728 in connection with the testing of guns.[12]

Euler's influence upon scientific life in Germany was not restricted to the Berlin Academy. He maintained a large correspondence with professors at numerous German universities and promoted the teaching of mathematical sciences and the preparation of university texts.

From his very first years in Berlin, Euler kept in regular working contact with the St. Petersburg Academy. This contact was interrupted only during military actions between Prussia and Russia in the course of the Seven Years' War—although even then not completely. Before his departure from the Russian capital, Euler was appointed an honorary member of the Academy and given an annual pension; on his part he pledged to carry out various assignments of the Academy and to correspond with it. During the twenty-five years in Berlin, Euler maintained membership in the St. Petersburg Academy *à tous les titres*, to quote N. Fuss. On its commission he finished the books on differential calculus and navigation begun before his departure for Berlin edited the mathematical section of the Academy journal; kept the Academy apprised, through his letters, of scientific and technological thought in Western Europe; bought books and scientific apparatus for the Academy; recommended subjects for scientific competitions and candidates to vacancies; and served as a mediator in conflicts between academicians.

Euler's participation in the training of Russian scientific personnel was of great importance, and he was frequently sent for review the works of Russian students and even members of the Academy. For example, in 1747 he praised most highly two articles of M.V. Lomonosov on physics and chemistry; and S.K. Kotelnikov, S. Y. Rumovski, and M. Sofronov studied in Berlin under his supervision for several years. Finally, Euler regularly sent memoirs to St. Petersburg. About half his articles were published there in Latin, and the other half appeared in French in Berlin.

During this period, Euler greatly increased the variety of his investigations. Competing with d'Alembert and Daniel Bernoulli, he laid the foundations of mathematical physics; and he was a rival of both A. Clairaut and d'Alembert in advancing the theory of lunar and planetary motion. At the same time, Euler elaborated the theory of motion of solids, created the mathematical apparatus of hydrodynamics, successfully developed the differential geometry of surfaces, and intensively studied optics, electricity, and magnetism. He also pondered such problems of technology as the construction of achromatic refractors, the perfection of J.A. Segner's hydraulic turbine, and the theory of toothed gearings.

During the Berlin period Euler prepared no fewer than 380 works, of which about 275 were published, including several lengthy books: a monograph on the calculus of variations (1744);[13] a fundamental work on calculation of orbits (1745);[14] the previously mentioned work on artillery and ballistics (1745); *Introductio in analysin infinitorum* (1748);[15] a treatise on shipbuilding and navigation, prepared in an early version in St. Petersburg (1749);[16] his first theory of lunar motion (1753);[17] and *Institutiones calculi differentialis* (1755).[18] The last three books were published at the expense of the St. Petersburg Academy. Finally, there was the treatise on the mechanics of solids, *Theoria motus corporum solidorum seu rigidorum* (1765).[19] The famous *Lettres à une princesse d'Allemagne sur divers sujets de physique et de philosophie*, which originated in lessons given by Euler to a relative of the Prussian king, was not published until Euler's return to St. Petersburg.[20] Written in an absorbing and popular manner, the book was an unusual success and ran to twelve editions in the original French, nine in English, six in German, four in Russian, and two in both Dutch and Swedish. There were also Italian, Spanish, and Danish editions.

In the 1740's and 1750's Euler took part in several philosophical and scientific arguments. In 1745 and after, there were passionate discussions about the monadology of Leibniz and of Christian Wolff. German intellectuals were divided according to their opinions on monadology. As Euler later wrote, every conversation ended in a discussion of monads. The Berlin Academy announced as the subject of a 1747 prize competition an exposé and critique of the system. Euler, who was close to Cartesian mechanical materialism in natural philosophy, was an ardent enemy of monadology, as was Maupertuis. It should be added that Euler, whose religious views were based on a belief in revelation, could not share the religion of reason which characterized Leibniz and Wolff. Euler stated his objections, which were grounded on arguments of both a physical and theological nature, in the pamphlet *Gedancken von den Elementen der Cörper*... (1746).[21] His composition caused violent debates, but the decision of the Academy gave the prize to Justi, author of a rather mediocre work against the theory of monads.

In 1751 a sensational new argument began when S. König published some critical remarks on Maupertuis's principle of least action (1744) and cited a letter of Leibniz in which the principle was, in König's opinion, formulated more precisely. Submitting to Maupertuis, the Berlin Academy rose to defend him and demanded that the original of Leibniz' letter (a copy had been sent to König from Switzerland) be presented. When it became clear that the original could not be found, Euler published, with the approval of the Academy, "Exposé concernant l'examen de la lettre de M. de Leibnitz" (1752),[22] where, among other things, he declared the letter a fake. The conflict grew critical when later in the same year Voltaire published his *Diatribe du docteur Akakia, médecin du pape*, defending König and making laughingstocks of both Maupertuis and Euler. Frederick rushed to the defense of Maupertuis, quarreling with his friend Voltaire and ordering the burning of the offensive pamphlet. His actions, however, did not prevent its dissemination throughout Europe. The argument touched not only on the pride of the principal participants but also on their general views: Maupertuis and, to a lesser degree, Euler interpreted the principle of least action theologically and teleologically; König was a follower of Wolff and Voltaire—the greatest ideologist of free thought.

Three other disputes in which Euler took part (all discussed below) were much more important for the development of mathematical sciences: his argument with d'Alembert on the problem of logarithms of negative numbers, the argument with d'Alembert and Daniel Bernoulli on the solution of the equation of a vibrating string, and Euler's polemics with Dollond on optical problems.

As mentioned earlier, after Maupertuis died in 1759, Euler managed the Berlin Academy, but under the direct supervision of the king. But relations between Frederick and Euler had long since spoiled. They differed sharply, not only in their views but in their tastes, treatment of men, and personal conduct. Euler's bourgeois manners and religious zeal were as unattractive to the king as the king's passion for bons mots and freethinking was to Euler. Euler cared little for poetry, which the king adored; Frederick was quite contemptuous of the higher realms of mathematics, which did not seem to him immediately practical. In spite of having no one to replace Euler as manager of the Academy, the king, nonetheless, did not intend to give him the post of president. In 1763 it became known that Frederick wanted to appoint d'Alembert, and Euler thus began to think of leaving Berlin. He wrote to G. F. Müller, secretary of the St. Petersburg Academy, which had tried earlier to bring him back to Russia. Catherine the Great then ordered the academicians to send Euler another offer.

D'Alembert's refusal to move permanently to Berlin postponed for a time the final decision on the matter. But during 1765 and 1766 grave conflicts over financial matters arose between Euler and Frederick, who interfered actively with Euler's management of the Academy after the Seven Years' War. The king thought Euler inexperienced in such matters and relied too much on the treasurer of the Academy. For half a year Euler pleaded for royal permission to leave, but the king, well-aware that the Academy would thus lose its best worker and principal force, declined to grant his request. Finally he had to consent and vented his annoyance in crude jokes about Euler. On 9 June 1766, Euler left Berlin, spent ten days in Warsaw at the invitation of Stanislas II, and arrived in St. Petersburg on 28 July. Euler's three sons returned to Russia also. Johann Albrecht became academician in the chair of physics in 1766 and permanent secretary of the Academy in 1769. Christoph, who had become an officer in Prussia, successfully resumed his military career, reaching the rank of major-general in artillery. Both his daughters also accompanied him.

Euler settled in a house on the embankment of the Neva, not far from the Academy. Soon after his return he suffered a brief illness, which left him almost completely blind in the left eye; he could not now read and could make out only outlines of large objects. He could write only in large letters with chalk and slate. An operation in 1771 temporarily restored his sight, but Euler seems not to have taken adequate care of himself and in a few days he was completely blind. Shortly before the operation, he had lost his house and almost all of his personal property in a fire, barely managing to rescue himself and his manuscripts. In November 1773 Euler's wife died, and three years later he married her half sister, Salome Abigail Gsell.

Euler's blindness did not lessen his scientific activity. Only in the last years of his life did he cease attending academic meetings, and his literary output even increased—almost half of his works were produced after 1765. His memory remained flawless, he carried on with his unrealized ideas, and he devised new plans. He naturally could not execute this immense work alone and was helped by active collaborators: his sons Johann Albrecht and Christoph; the academicians W. L. Krafft and A. J. Lexell; and two new young disciples, adjuncts N. Fuss, who was invited in 1772 from Switzerland, and M. E. Golovin, a nephew of Lomonosov. Sometimes Euler simply dictated his works; thus, he dictated to a young valet, a tailor by profession, the two-volume *Vollstàndige Anleitung zur Algebra* (l770),[23] first published in Russian translation.

But the scientists assisting Euler were not mere secretaries; he discussed the general scheme of the works with them, and they developed his ideas, calculated tables, and sometimes compiled examples. The enormous, 775-page *Theoria motuum lunae*... (1772)[24] was thus completed with the help of Johann Albrecht, Krafft, and Lexell—all of whom are credited on the title page. Krafft also helped Euler with the three-volume *Dioptrica* (l769-1771).[25] Fuss, by his own account, during a seven-year period prepared 250 memoirs, and Golovin prepared seventy. Articles written by Euler in his later years were generally concise and particular. For example, the fifty-six works prepared during 1776 contain about the same number of pages (1,000) as the nineteen works prepared in 1751.

Besides the works mentioned, during the second St. Petersburg period Euler published three volumes of *Institutiones calculi integralis* (1768–l770),[26] the principal parts of which he had finished in Berlin, and an abridged edition of *Scientia navalis—Théorie complette de la construction et de la manoeuvre des vaisseaux* (1773).[27] The last, a manual for naval cadets, was soon translated into English, Italian, and Russian, and Euler received for it large sums from the Russian and French governments.

The mathematical apparatus of the *Dioptrica* remained beyond the practical opticist's understanding; so Fuss devised, on the basis of this work, the *Instruction détaillée pour porter les lunettes de toutes les différentes espéces au plus haut degré de perfection dont elles sont susceptibles*... (1774).[28] Fuss also aided Euler in preparing the *Éclaircissemens sur les érablissemens publics*... (l776),[29] which was very important in the development of insurance; many companies used its methods of solution and its tables.

Euler continued his participation in other functions of the St. Petersburg Academy. Together with Johann Albrecht he was a member of the commission charged in 1766 with the management of the Academy. Both resigned their posts on the commission in 1774 because of a difference of opinion between them and the director of the Academy, Count V. G. Orlov, who actually managed it.

On 18 September 1783 Euler spent the first half of the day as usual. He gave a mathematics lesson to one of his grandchildren, did some calculations with chalk on two boards on the motion of balloons; then discussed with Lexell and Fuss the recently discovered planet Uranus. About five o'clock in the afternoon he suffered a brain hemorrhage and uttered only "I am dying," before he lost consciousness. He died about eleven o'clock in the evening.

Soon after Euler's death eulogies were delivered by Fuss at a meeting of the St. Petersburg Academy[30] and by Condorcet at the Paris Academy of Sciences.[31] Euler was buried at the Lutheran Smolenskoye cemetery in St. Petersburg, where in 1837 a massive monument was erected at his grave, with the inscription, "Leonhardo Eulero Academia Petropolitana." In the autumn of 1956 Euler's remains and the monument were transferred to the necropolis of Leningrad.

Euler was a simple man, well disposed and not given to envy. One can also say of him what Fontenelle said of Leibniz: "He was glad to observe the flowering in other people's gardens of plants whose seeds he provided."

**Mathematics** . Euler was a geometer in the wide sense in which the word was used during the eighteenth century. He was one of the most important creators of mathematical science after Newton. In his work, mathematics was closely connected with applications to other sciences, to problems of technology, and to public life. In numerous cases he elaborated mathematical methods for the direct solution of problems of mechanics and astronomy, physics and navigation, geography and geodesy, hydraulics and ballistics, insurance and demography. This practical orientation of his work explains his tendency to prolong his investigations until he had derived a convenient formula for calculation or an immediate solution in numbers or a table. He constantly sought algorithms that would be simple to use in calculation and that would also assure sufficient accuracy in the results.

But just as his friend Daniel Bernoulli was first of all a physicist, Euler was first of all a mathematician. Bernoulli's thinking was preeminently physical; he tried to avoid mathematics whenever possible, and once having developed a mathematical device for the solution of some physical problem, he usually left it without further development. Euler, on the other hand, attempted first of all to express a physical problem in mathematical terms; and having found a mathematical idea for solution, he systematically developed and generalized it. Thus, Euler's brilliant achievements in the field are explained by his regular elaboration of mathematics as a single whole. Bernoulli was not especially attracted by more abstract problems of mathematics; Euler, on the contrary, was very much carried away with the theory of numbers. All this is manifest in the distribution of Euler's works on various sciences: twenty-nine volumes of the *Opera omnia* (see BIBLIOGRAPHY [1]) pertain to pure mathematics.

In Euler's mathematical work, first place belongs to analysis, which at the time was the most pressing need in mathematical science; seventeen volumes of the *Opera omnia* are in this area. Thus, in principle, Euler was an analyst. He contributed numerous particular discoveries to analysis, systematized its exposition in his classical manuals, and, along with all this, contributed immeasurably to the founding of several large mathematical disciplines: the calculus of variations, the theory of differential equations, the elementary theory of functions of complex variables, and the theory of special functions.

Euler is often characterized as a calculator of genius, and he was, in fact, unsurpassed in formal calculations and transformations and was even an outstanding calculator in the elementary sense of the word. But he also was a creator of new and important notions and methods, the principal value of which was in some cases properly understood only a century or more after his death. Even in areas where he, along with his contemporaries, did not feel at home, his judgment came, as a rule, from profound intuition into the subject under study. His findings were intrinsically capable of being grounded in the rigorous mode of demonstration that became obligatory in the nineteenth and twentieth centuries. Such standards were not, and could not be, demanded in the mathematics of the eighteenth century.

It is frequently said that Euler saw no intrinsic impossibility in the deduction of mathematical laws from a very limited basis in observation; and naturally he employed methods of induction to make empirical use of the results he had arrived at through analysis of concrete numerical material. But he himself warned many times that an incomplete induction serves only as a heuristic device, and he never passed off as finally proved truths the suppositions arrived at by such methods.

Euler introduced many of the present conventions of mathematical notation: the symbol $e$ to represent the base of the natural system of logarithms (1727, published 1736); the use of letter $f$ and of parentheses for a function $f([x/a] + c)$ (1734, published 1740); the modern signs for trigonometric functions (1748); the notation $fn$ for the sum of divisors of the number $n$ (1750); notations for finite differences, $\Delta y$, $\Delta^2 y$, etc., and for the sum $\Sigma$(1755); and the letter $i$ for , published 1794).

Euler had only a few immediate disciples, and none of them was a first-class scientist. On the other hand, according to Laplace, he was a tutor of all the mathematicians of his time. In mathematics the eighteenth century can fairly be labeled the Age of Euler, but his influence upon the development of mathematical sciences was not restricted to that period. The work of many outstanding nineteenth-century mathematicians branched out directly from the works of Euler.

Euler was especially important for the development of science in Russia. His disciples formed the first scientific mathematical school in the country and contributed to the rise of mathematical education. One can trace back to Euler numerous paths from Chebyshev's St. Petersburg mathematical school.

[In the following, titles of articles are not, as a rule, cited; dates in parentheses signify the year of publication.]

**Theory of Numbers** . Problems of the theory of numbers had attracted mathematicians before Euler. Fermat, for example, established several remarkable arithmetic theorems but left almost no proofs. Euler laid the foundations of number theory as a true science.

A large series of Euler's works is connected with the theory of divisibility. He proved by three methods Fermat's lesser theorem, the principal one in the field (1741, 1761, 1763); he suggested with the third proof an important generalization of the theorem by introducing Euler's function $\phi(n)$, denoting the number of positive integers less than $n$ which are relatively prime to $n$: the difference $a^{\phi n}-1$ is divisible by n if $a$ is relatively prime to $n$. Elaborating related ideas, Euler came to the theory of *n-ic* residues (1760). Here his greatest discovery was the law of quadratic reciprocity (1783), which, however, he could not prove. Euler's discovery went unnoticed by his contemporaries, and the law was rediscovered, but incompletely proved, by A. M. Legendre (1788). Legendre was credited with it until Chebyshev pointed out Euler's priority in 1849. The complete proof of the law was finally achieved by Gauss (1801). Gauss, Kummer, D. Hilbert, E. Artin, and others extended the law of reciprocity to various algebraic number fields; the most general law of reciprocity was established by I. R. Shafarevich (1950).

Another group of Euler's works, in which he extended Fermat's studies on representation of prime numbers by sums of the form $m x^2+n y^2$, where m, $n$, $x$, and y are positive integers, led him to the discovery of a new efficient method of determining whether a given large number N is prime or composite (1751, *et seq*.). These works formed the basis for the general arithmetic theory of binary quadratic forms developed by Lagrange and especially by Gauss.

Euler also contributed to so-called Diophantine analysis, that is, to the solution, in integers or in rational numbers, of indeterminate equations with integer coefficients. Thus, by means of continued fractions, which he had studied earlier (1744, *et seq*.), he gave (1767) a method of calculation of the smallest integer solution of the equation $x^2-d y^2=1$ ($d$ being a positive nonsquare integer). This had been studied by Fermat and Wallis and even earlier by scientists of India and Greece. A complete investigation of the problem was soon undertaken by Lagrange. In 1753 Euler proved the impossibility of solving $x^3+y^3=z^3$ in which $x, y$, and $z$ are integers, $xyz \neq 0$ (a particular case of Fermat's last theorem); his demonstration, based on the method of infinite descent and using complex numbers of the form , is thoroughly described in his *Vllstàndige Anleitung zur Algebra*, the second volume of which (1769) has a large section devoted to Diophantine analysis.

In all these cases Euler used methods of arithmetic and algebra, but he was also the first to use analytical methods in number theory. To solve the partition problem posed in 1740 by P. Naudé, concerning the total number of ways the positive integer *n* is obtainable as a sum of positive integers $m < n$, Euler used the expansions of certain infinite products into a power series whose coefficients give the solution (1748). In particular, in the expansion the right-hand series is one of theta functions, introduced much later by C. Jacobi in his theory of elliptic functions. Earlier, in 1737, Euler had deduced the famous identity where the sum extends over all positive integers *n* and the product over all primes *p* (1744), the left-hand side is what Riemann later called the zeta-function ζ(ς)

Using summation of divergent series and induction, Euler discovered in 1749 (1768) a functional equation involving $\zeta(s)$, $\zeta(1-s)$, and $\Gamma(s)$, which was rediscovered and established by Riemann, the first scientist to define the zeta-function also for complex values of the argument. In the nineteenth and twentieth centuries, the zeta-function became one of the principal means of analytic number theory, particularly in the studies of the laws of distribution of prime numbers by Dirichlet, Chebyshev, Riemann, Hadamard, de la Vallèe-Poussin, and others.

Finally, Euler studied mathematical constants and formulated important problems relevant to the theory of transcendental numbers. His expression of the number e in the form of a continued fraction (1744) was used by J. H. Lambert (1768) in his demonstration of irrationality of the numbers *e* and π. F. Lindemann employed Euler's formula Ln(-1) = πί (discovered as early as 1728) to prove that is transcendental (1882). The hypothesis of the transcendence of $a^b$, where *a* is any algebraic number ≠0,1 and b is any irrational algebraic number—formulated by D. Hilbert in 1900 and proved by A. Gelfond in 1934—presents a generalization of Euler's corresponding supposition about rational-base logarithms of rational numbers (1748).

**Algebra** . When mathematicians of the seventeenth century formulated the fundamental theorem that an algebraic equation of degree *n* with real coefficients has *n* roots, which could be imaginary, it was yet unknown whether the domain of imaginary roots was restricted to numbers of the form a + bi, which, following Gauss, are now called complex numbers. Many mathematicians thought that there existed imaginary quantities of another kind. In his letters to Nikolaus I Bernoulli and to Goldbach (dated 1742), Euler stated for the first time the theorem that every algebraic polynomial of degree *n* with real coefficients may be resolved into real linear or quadratic factors, that is, possesses *n* roots of the form $a + bi$ (1743). The theorem was proved by d'Alembert (1748) and by Euler himself (1751). Both proofs, quite different in ideas, had omissions and were rendered more precise during the nineteenth century.

Euler also aspired—certainly in vain—to find the general form of solution by radicals for equations of degree higher than the fourth (1738, 1764). He elaborated approximating methods of solutions for numerical equations (1748) and studied the elimination problem. Thus, he gave the first proof of the theorem, which was known to Newton, that two algebraic curves of degrees *m* and *n*, respectively, intersect in *mn* points (1748, 1750). It should be added that Euler's *Vollständige Anleitung zur Algebra*, published in many editions in English, Dutch, Italian, French, and Russian, greatly influenced nineteenth- and twentieth-century texts on the subject.

**Infinite Series** . In Euler's works, infinite series, which previously served mainly as an auxiliary means for solving problems, became a subject of study. One example, his investigation of the zeta-function, has already been mentioned. The point of departure was the problem of summation of the reciprocals of the squares of the integers

which had been vainly approached by the Bernoulli brothers, Stirling, and other outstanding mathematicians. Euler solved in 1735 a much more general problem and demonstrated that for any even integer number $2k > 0$,

$\zeta(2k)=a_{2k}\pi^{s\,k}$,

where $a_{2k}$ are rational numbers (1740), expressed through coefficients of the Euler-Maclaurin summation formula (1750) and, consequently, through Bernoulli numbers (1755). The problem of the arithmetic nature of ζ(2k + 1) remains unsolved.

The summation formula was discovered by Euler no later than 1732 (1738) and demonstrated in 1735 (1741); it was independently discovered by Maclaurin no later than 1738 (1742). The formula, one of the most important in the calculus of finite differences, represents the partial sum of a series, , by another infinite series involving the integral and the derivatives of the general term *u(n)*. Later Euler expressed the coefficients of the latter series through Bernoulli numbers (1755). Euler knew that although this infinite series generally diverges, its partial sums under certain conditions might serve as a brilliant means of approximating the calculations shown by James Stirling (1730) in a particular case of

By means of the summation formula, Euler in 1735 calculated (1741) to sixteen decimal places the value of Euler's constant,

$C = 0.57721566...$,

belonging to an asymptotic formula,

which he discovered in 1731 (1738).

The functions studied in the eighteenth century were, with rare exceptions, analytic, and therefore Euler made great use of power series. His special merit was the introduction of a new and extremely important class of trigonometric Fourier series. In a letter to Goldbach (1744), he expressed for the first time an algebraic function by such a series (1755),

He later found other expansions (1760), deducing in 1777 a formula of Fourier coefficients for expansion of a given function into a series of cosines on the interval (0,π), pointing out that coefficients of expansion into a series of sines could be deduced analogously (1798). Fourier, having no knowledge of Euler's work, deduced in 1807 the same formulas. For his part, Euler did not know that coefficients of expansion into a series of cosines had been given by Clairaut in 1759.

Euler also introduced expansion of functions into infinite products and into the sums of elementary fractions, which later acquired great importance in the general theory of analytic functions. Numerous methods of transformation of infinite series, products, and continued fractions into one another are also his.

Eighteenth-century mathematicians distinguished convergent series from divergent series, but the general theory of convergence was still missing. Algebraic and analytic operations on infinite series were similar to those on finite polynomials, without any restrictions. It was supposed that identical laws operate in both cases. Several tests of convergence already known found almost no application. Opinions, however, differed on the problem of admissibility of divergent series. Many mathematicians were radically against their employment. Euler, sure that important correct results might be arrived at by means of divergent series, set about the task of establishing the legitimacy of their application. With this aim, he suggested a new, wider definition of the concept of the sum of a series, which coincides with the traditional definition if the series converges; he also suggested two methods of summation (1755). Precise grounding and further development of these fruitful ideas were possible only toward the end of the nineteenth century and the beginning of the twentieth century.[32]

**The Concept of Function** . Discoveries in the field of analysis made in the middle of the eighteenth century (many of them his own) were systematically summarized by Euler in the trilogy *Introductio in analysin infinitorum* (1748),[15]*Institutiones calculi differentialis* (1755),[18] and *Institutiones calculi integralis* (1768-1770). The books are still of interest, especially the first volume of the *Introductio*. Many of the problems considered there, however, are now so far developed that knowledge of them is limited to a few specialists, who can trace in the book the development of many fruitful methods of analysis.

In the *Introductio* Euler presented the first clear statement of the idea that mathematical analysis is a science of functions; and he also presented a more thorough investigation of the very concept of function. Defining function as an analytic expression somehow composed of variables and constants—following in this respect Johann I Bernoulli (1718)— Euler defined precisely the term "analytic expression": functions are produced by means of algebraic operations, and also of elementary and other transcendental operations, carried out by integration. Here the classification of functions generally used today is also given; Euler speaks of functions defined implicitly and by parametric representation. Further on he states his belief, shared by other mathematicians, that all analytic expressions might be given in the form of infinite power series or generalized power series with fractional or negative exponents. Thus, functions studied in mathematical analysis generally are analytic functions with some isolated singular points. Euler's remark that functions are considered not only for real but also for imaginary values of independent variables was very important.

Even at that time, however, the class of analytic functions was insufficient for the requirements of analysis and its applications, particularly for the solution of the problem of the vibrating string. Here Euler encountered "arbitrary" functions, geometrically represented in piecewise smooth plane curves of arbitrary form—functions which are, generally speaking, nonanalytic (1749). The problem of the magnitude of the class of functions applied in mathematical physics and generally in analysis and the closely related problem of the possibility of analytic expression of nonanalytic functions led to a lengthy polemic involving many mathematicians, including Euler, d'Alembert, and Daniel Bernoulli. One of the results of this controversy over the problem of the vibrating string was the general arithmetical definition of a function as a quantity whose values somehow change with the changes of independent variables; the definition was given by Euler in *Institutiones calculi differentialis*.[18] He had, however, already dealt with the interpretation of a function as a correspondence of values in his *introductio*.

**Elementary Functions** . The major portion of the first volume of the *Introductio* is devoted to the theory of elementary functions, which is developed by means of algebra and of infinite series and products. Concepts of infinitesimal and infinite quantity are used, but those of differential and integral calculus are lacking. Among other things, Euler here for the first time described the analytic theory of trigonometric functions and gave a remarkably simple, although nonrigorous, deduction of Moivre's formula and also of his own (1743),

$$e^{\pm xi} = cos\ x \pm i\ sin\ x.$$

This was given earlier by R. Cotes (1716) in a somewhat different formulation, but it was widely used only by Euler. The logarithmic function was considered by Euler in the *Introductio* only for the positive independent variable. However, he soon published his complete theory of logarithms of complex numbers (1751)—which some time before had ended the arguments over logarithms of negative numbers between Leibniz and Johann Bernoulli and between d'Alembert and Euler himself in their correspondence (1747-1748). Euler had come across the problem (1727-1728) when he discussed in his correspondence with Johann I Bernoulli the problem of the graphics of the function $y = (-1)^x$ and arrived at the equality $\ln(-1)= \pi i$.

**Functions of a Complex Variable** . The study of elementary functions brought d'Alembert (1747-1748) and Euler (1751) to the conclusion that the domain of complex numbers is closed (in modern terms) with regard to all algebraic and transcendental operations. They both also made early advances in the general theory of analytic functions. In 1752 d'Alembert, investigating problems of hydrodynamics, discovered equations connecting the real and imaginary parts of an analytic function $u(x,y) + iv(x,y)$. In 1777 Euler deduced the same equations, from general analytical considerations, developing a new method of calculation of definite integrals $\int f(z)\, dz$ by means of an imaginary substitution

$z = x + iy$

(1793, 1797). He thus discovered (1794) that

Euler also used analytic functions of a complex variable, both in the study of orthogonal trajectories by means of their conformal mapping (1770) and in his works on cartography (1778). (The term *projectio conformis* was introduced by a St. Petersburg academician, F. T. Schubert [1789].) All of these ideas were developed in depth in the elaboration of the general theory of analytic functions by Cauchy (1825) and Riemann (1854), after whom the above-cited equations of d'Alembert and Euler are named.

Although Euler went from numbers of the form $x + iy$ to the point $u(x,y)$ and back, and used a trigonometric form $r(\cos \phi + i \sin\phi)$, he saw in imaginary numbers only convenient notations void of real meaning. A somewhat less than successful attempt at geometric interpretation undertaken by H. Kühn (1753) met with sharp critical remarks from Euler.

**Differential and Integral Calculus** . Both branches of infinitesimal analysis were enriched by Euler's numerous discoveries. Among other things in the *Institutiones calculi differentialis*, he thoroughly elaborated formulas of differentiation under substitution of variables; revealed his theorem on homogeneous functions, stated for $f(x,y)$ as early as 1736; proved the theorem of Nikolaus I Bernoulli (1721) that for $z = f(x,y)$

deduced the necessary condition for the exact differential of $f(x,y)$; applied Taylor's series to finding extrema of $f(x)$; and investigated extrema of $f(x,y)$, inaccurately formulating, however, sufficient conditions.

The first two chapters of the *Institutiones* are devoted to the elements of the calculus of finite differences. Euler approached differential calculus as a particular case, we would say a limiting case, of the method of finite differences used when differences of the function and of the independent variable approach zero. During the eighteenth century it was often said against differential calculus that all its formulas were incorrect because the deductions were based on the principle of neglecting infinitely small summands, e.g., on equalities of the kind $a + \alpha = a$, where $\alpha$ is infinitesimal with respect to $a$. Euler thought that such criticism could be obviated only by supposing all infinitesimals and differentials equal to zero, and therefore he elaborated an original calculus of zeroes. This concept, although not contradictory in itself, did not endure because it proved insufficient in many problems; a strict grounding of analysis was possible if the infinitesimals were interpreted as variables tending to the limit zero.

The methods of indefinite integration in the *Institutiones calculi integralis* (I, 1768) are described by Euler in quite modern fashion and in a detail that practically exhausts all the cases in which the result of integration is expressible in elementary functions. He invented many of the methods himself; the expression "Euler substitution" (for rationalization of certain irrational differentials) serves as a reminder of the fact. Euler calculated many difficult definite integrals, thus laying the foundations of the theory of special functions. In 1729, already studying interpolation of the sequence 1!, 2!,..., n!,..., he introduced Eulerian integrals of the first and second kind (Legendre's term), today called the beta-and gammafunctions (1738). He later discovered a number of their properties.

Particular cases of the beta-function were first considered by Wallis in 1656. The functions **B** and Γ, together with the zeta-function and the so-called Bessel functions (see below), are among the most important transcendental functions. Euler's main contribution to the theory of elliptic integrals was his discovery of the general addition theorem (1768). Finally, the theory of multiple integrals also goes back to Euler; he introduced double integrals and established the rule of substitution (1770).

**Differential Equations** . The *Institutiones calculi integralis* exhibits Euler's numerous discoveries in the theory of both ordinary and partial differential equations, which were especially useful in mechanics.

Euler elaborated many problems in the theory of ordinary linear equations: a classical method for solving reduced linear equations with constant coefficients, in which he strictly distinguished between the general and the particular integral (1743); works on linear systems, conducted simultaneously with d'Alembert (1750); solution of the general linear equation of order $n$ with constant coefficients by reduction to the equation of the same form of order $n$ - 1 (1753). After 1738 he successfully applied to second-order linear equations with variable coefficients a method that was highly developed in the nineteenth century; this consisted of the presentation of particular solutions in the form of generalized power series. Another Eulerian device, that of expressing solutions by definite integrals that depend on a parameter (1763), was extended by Laplace to partial differential equations (1777).

One can trace back to Euler (1741) and Daniel Bernoulli the method of variation of constants later elaborated by Lagrange (1777). The method of an integrating factor was also greatly developed by Euler, who applied it to numerous classes of first-order differential equations (1768) and extended it to higher-order equations (1770). He devoted a number of articles to the Riccati equation, demonstrating its involvement with continued fractions (1744). In connection with his works on the theory of lunar motion, Euler created the widely used device of approximating the solution of the equation $dy/dx = f(x,y)$, with initial condition $x = x_0$, $y = y_0$ (1768), extending it to second-order equations (1769). This Euler method of open polygons was used by Cauchy in the 1820's to demonstrate the existence theorem for the solution of the above-mentioned equation (1835, 1844). Finally, Euler discovered tests for singular solutions of first-order equations (1768).

Among the large cycle of Euler's works on partial differential equations begun in the middle of the 1730's with the study of separate kinds of first-order equations, which he had encountered in certain problems of geometry (1740), the most important are the studies on second-order linear equations—to which many problems of mathematical physics may be reduced. First was the problem of small plane vibrations of a string, the wave equation originally solved by d'Alembert with the so-called method of characteristics. Given a general solution expressible as a sum of two arbitrary functions, the initial conditions and the boundary conditions of the problem admitted of arriving at solutions in concrete cases (1749). Euler immediately tested this method of d'Alembert's and further elaborated it, eliminating unnecessary restrictions imposed by d'Alembert upon the initial shape and velocity of the string (1749). As previously mentioned, the two mathematicians engaged in an argument which grew more involved when Daniel Bernoulli asserted that any solution of the wave equation might be expressed by a trigonometric series (1755). D'Alembert and Euler agreed that such a solution could not be sufficiently general. The discussion was joined by Lagrange, Laplace, and other mathematicians of great reputation and lasted for over half a century; not until Fourier (1807, 1822) was the way found to the correct formulation and solution of the problem. Euler later developed the method of characteristics more thoroughly (1766, 1767).

Euler encountered equations in other areas of what became mathematical physics: in hydrodynamics; in the problem of vibrations of membranes, which he reduced to the so-called Bessel equation and solved (1766) by means of the Bessel functions $J_n(x)$; and in the problem of the motion of air in pipes (1772). Some classes of equations studied by Euler for velocities close to or surpassing the velocity of sound continue to figure in modern aerodynamics.

**Calculus of Variations** . Starting with several problems solved by Johann and Jakob Bernoulli, Euler was the first to formulate the principal problems of the calculus of variations and to create general methods for their solution. In *Methodus inveniendi lineas curvas...*[13] he systematically developed his discoveries of the 1730's (1739, 1741). The very title of the work shows that Euler widely employed geometric representations of functions as flat curves. Here he introduced, using different terminology, the concepts of function and variation and distinguished between problems of absolute extrema and relative extrema, showing how the latter are reduced to the former. The problem of the absolute extremum of the function of several independent variables, where $F$ is the given and $y(x)$ the desired minimizing or maximizing function, is treated as the limiting problem for the ordinary extremum of the function where $x_k = a+k\ \Delta x$, $\Delta x = (b - a)/n$, $k = 0, 1,..., n$ (and$\rightarrow\infty$). Thus Euler deduced the differential equation named after him to which the function $y(x)$ should correspond; this necessary condition was generalized for the case where F involves the derivatives $y', y'',..., y^n$. In this way the solution of a problem in the calculus of variations might always be reduced to integration of a differential equation. A century and a half later the situation had changed. The direct method imagined by Euler, which he had employed only to obtain his differential equation, had (together with similar methods) acquired independent value for rigorous or approximate solution of variational problems and the corresponding differential equations.

In the mid-1750's, after Lagrange had created new algorithms and notations for the calculus of variations, Euler abandoned the former exposition and gave instead a detailed and lucid exposition of Lagrange's method, considering it a new calculus—which he called variational (1766). He applied the calculus of variations to problems of extreme values of double integrals with constant limits in volume III of the *Institutiones calculi integralis* (1770); soon thereafter he suggested still another method of exposition of the calculus, one which became widely used.

**Geometry** . Most of Euler's geometrical discoveries were made by application of the methods of algebra and analysis. He gave two different methods for an analytical exposition of the system of spherical trigonometry (1755, 1782). He showed how the trigonometry of spheroidal surfaces might be applied to higher geodesy (1755). In volume II of the *Introductio* he surpassed his contemporaries in giving a consistent algebraic development of the theory of second-order curves, proceeding from their general equation (1748). He constituted the theory of third-order curves by analogy. But Euler's main achievement was that for the first time he studied thoroughly the general equation of second-order surfaces, applying Euler angles in corresponding transformations.

Euler's studies of the geodesic lines on a surface are prominent in differential geometry; the problem was pointed out to him by Johann Bernoulli (1732, 1736, and later). But still more important were his pioneer investigations in the theory of surfaces, from which Monge and other geometers later proceeded. In 1763 Euler made the first substantial advance in the study of the curvature of surfaces; in particular, he expressed the curvature of an arbitrary normal section by principal curvatures (1767). He went on to study developable surfaces, introducing Gaussian coordinates (1772), which became widely used in the nineteenth century. In a note written about 1770 but not published until 1862 Euler discovered the necessary condition for applicability of surfaces that was independently established by Gauss (1828). In 1775 Euler successfully renewed elaboration of the general theory of space curves (1786), beginning where Clairaut had left off in 1731.

Euler was also the author of the first studies on topology. In 1735 he gave a solution to the problem of the seven bridges of Königsberg: the bridges, spanning several arms of a river, must all be crossed without recrossing any (1741). In a letter to Goldbach (1750), he cited (1758) a number of properties of polyhedra, among them the following: the number of vertices, $S$, edges, $A$, and sides, $H$, of a polyhedron are connected by an equality $S - A + H = 2$. A hundred years later it was discovered that the theorem had been known to Descartes. The Euler characteristic $S - A + H$ and its generalization for multidimensional complexes as given by H. Poincaré is one of the principal invariants of modern topology.

**Mechanics** . In an introduction to the *Mechanica* (1736) Euler outlined a large program of studies embracing every branch of the science. The distinguishing feature of Euler's investigations in mechanics as compared to those of his predecessors is the systematic and successful application of analysis. Previously the methods of mechanics had been mostly synthetic and geometrical; they demanded too individual an approach to separate problems. Euler was the first to appreciate the importance of introducing uniform analytic methods into mechanics, thus enabling its problems to be solved in a clear and direct way. Euler's concept is manifest in both the introduction and the very title of the book, *Mechanica sine motus scientia analytice exposita*.

This first large work on mechanics was devoted to the kinematics and dynamics of a point-mass. The first volume deals with the free motion of a point-mass in a vacuum and in a resisting medium; the section on the motion of a point-mass under a force directed to a fixed center is a brilliant analytical reformulation of the corresponding section of Newton's *Principia*; it was sort of an introduction to Euler's further works on celestial mechanics. In the second volume, Euler studied the constrained motion of a point-mass; he obtained three equations of motion in space by projecting forces on the axes of a moving trihedral of a trajectory described by a moving point, i.e., on the tangent, principal normal, and binormal. Motion in the plane is considered analogously. In the chapter on the motion of a point on a given surface, Euler solved a number of problems of the differential geometry of surfaces and of the theory of geodesics.

*The Theoria motus corporum solidorum*[19] published almost thirty years later (1765), is related to the *Mechanica*. In the introduction to this work, Euler gave a new exposition of punctual mechanics and followed Maclaurin's example (1742) in projecting the forces onto the axes of a fixed orthogonal rectilinear system. Establishing that the instantaneous motion of a solid body might be regarded as composed of rectilinear translation and instant rotation, Euler devoted special attention to the study of rotatory motion. Thus, he gave formulas for projections of instantaneous angular velocity on the axes of coordinates (with application of Euler angles), and framed dynamical differential equations referred to the principal axes of inertia, which determine this motion. Special mention should be made of the problem of motion of a heavy solid body about a fixed point, which Euler solved for a case in which the center of gravity coincides with the fixed point. The law of motion in such a case is, generally speaking, expressed by means of elliptic integrals. Euler was led to this problem by the study of precession of the equinoxes and of the nutation of the terrestrial axis (1751).[33] Other cases in which the differential equations of this problem can be integrated were discovered by Lagrange (1788) and S. V. Kovalevskaya (1888). Euler considered problems of the mechanics of solid bodies as early as the first St. Petersburg period.

In one of the two appendixes to the *Methodus*...[13] Euler suggested a formulation of the principle of least action for the case of the motion of a point under a central force: the trajectory described by the point minimizes the integral *ƒ mu ds*. Maupertuis had stated at nearly the same time the principle of least action in a much more particular form. Euler thus laid the mathematical foundation of the numerous studies on variational principles of mechanics and physics which are still being carried out.

In the other appendix to the *Methodus*, Euler, at the insistence of Daniel Bernoulli, applied the calculus of variations to some problems of the theory of elasticity, which he had been intensively elaborating since 1727. In this appendix, which was in fact the first general work on the mathematical theory of elasticity, Euler studied bending and vibrations of elastic bands (either homogeneous or nonhomogeneous) and of a plate under different conditions; considered nine types of elastic curves; and deduced the famous Euler buckling formula, or Euler critical load, used to determine the strength of columns.

**Hydromechanics** . Euler's first large work on fluid mechanics was *Scientia novalis*. Volume I contains a general theory of equilibrium of floating bodies including an original elaboration of problems of stability and of small oscillations in the neighborhood of an equilibrium position. The second volume applies general theorems to the case of a ship.

From 1753 to 1755 Euler elaborated in detail an analytical theory of fluid mechanics in three classic memoirs—"Principes généraux de l'état d'équilibre des fluides" "Principes généraux du mouvement des fluides"; and "Continuation des recherches sur la théorie du mouvement des fluides"—all published simultaneously (1757).[34] Somewhat earlier (1752) the "Principia motus fluidorum" was written; it was not published, however, until 1761.[35] Here a system of principal formulas of hydrostatics and hydrodynamics was for the first time created; it comprised the continuity equation for liquids with constant density; the velocity-potential equation (usually called after Laplace); and the general Euler equations for the motion of an incompressible liquid, gas, etc. As has generally been the case in mathematical physics, the main innovations were in the application of partial differential equations to the problems. At the beginning of the "Continuation des recherches" Euler emphasized that he had reduced the whole of the theory of liquids to two analytic equations and added:

However sublime are the researches on fluids which we owe to the Messrs. Bernoulli, Clairaut and d'Alembert, they flow so naturally from my two general formulae that one cannot sufficiently admire this accord of their profound meditations with the simplicity of the principles from which I have drawn my two equations, and to which I was led immediately by the first axioms of mechanics.[36]

Euler also investigated a number of concrete problems on the motion of liquids and gases in pipes, on vibration of air in pipes, and on propagation of sound. Along with this, he worked on problems of hydrotechnology, discussed, in part, above. Especially remarkable were the improvements he introduced into the design of a hydraulic machine imagined by Segner in 1749 and the theory of hydraulic turbines, which he created in accordance with the principle of action and reaction (1752–1761).[37]

**Astronomy** . Euler's studies in astronomy embraced a great variety of problems: determination of the orbits of comets and planets by a few observations, methods of calculation of the parallax of the sun, the theory of refraction, considerations on the physical nature of comets, and the problem of retardation of planetary motions under the action of cosmic ether. His most outstanding works, for which he won many prizes from the Paris Académie des Sciences, are concerned with celestial mechanics, which especially attracted scientists at that time.

The observed motions of the planets, particularly of Jupiter and Saturn, as well as the moon, were evidently different from the calculated motions based on Newton's theory of gravitation. Thus, the calculations of Clairaut and d'Alembert (1745) gave the value of eighteen years for the period of revolution of the lunar perigee, whereas observations showed this value to be nine years. This caused doubts about the validity of Newton's system as a whole. For a long time Euler joined these scientists in thinking that the law of gravitation needed some corrections. In 1749 Clairaut established that the difference between theory and observation was due to the fact that he and others solving the corresponding differential equation had restricted themselves to the first approximation. When he calculated the second approximation, it was satisfactorily in accordance with the observed data. Euler did not at once agree. To put his doubts at rest, he advised the St. Petersburg Academy to announce a competition on the subject. Euler soon determined that Clairaut was right, and on Euler's recommendation his composition received the prize of the Academy (1752). Euler was still not completely satisfied, however. In 1751 he had written his own *Theoria motus lunae exhibens omnes ejus inaequalitates* (published in 1753), in which he elaborated an original method of approximate solution to the three-body problem, the so-called first Euler lunar theory. In the appendix he described another method which was the earliest form of the general method of variation of elements. Euler's numerical results also conformed to Newton's theory of gravitation.

The first Euler lunar theory had an important practical consequence: T. Mayer, an astronomer from Göttingen, compiled, according to its formulas, lunar tables (1755) that enabled the calculation of the position of the moon and thus the longitude of a ship with an exactness previously unknown in navigation. The British Parliament had announced as early as 1714 a large cash prize for the method of determination of longitude at sea with error not to exceed half a degree, and smaller prizes for less exact methods. The prize was not awarded until 1765; £ 3,000 went to Mayer's widow and £300 to Euler for his preliminary theoretical work. Simultaneously a large prize was awarded to J. Harrison for his construction of a more nearly perfect chronometer. Lunar tables were included in all nautical almanacs after 1767, and the method was used for about a century.

From 1770 to 1772 Euler elaborated his second theory of lunar motion, which he published in the *Theoria motuum lunae, nova methodo pertractata* (1772).[24] For various reasons, the merits of the new method could be correctly appreciated only after G. W. Hill brilliantly developed the ideas of the composition in 1877-1888.

Euler devoted numerous works to the calculation of perturbations of planetary orbits caused by the mutual gravitation of Jupiter and Saturn (1749, 1769) as well as of the earth and the other planets (1771). He continued these studies almost to his death.

**Physics** . Euler's principal contribution to physics consisted in mathematical elaboration of the problems discussed above. He touched upon various physical problems which would not yield to mathematical analysis at that time. He aspired to create a uniform picture of the physical world. He had been, as pointed out earlier, closer to Cartesian natural philosophy than to Newtonian, although he was not a direct representative of Cartesianism. Rejecting the notion of empty space and the possibility of action at a distance, he thought that the universe is filled up with ether—a thin elastic matter with extremely low density, like super-rarified air. This ether contains material particles whose main property is impenetrability. Euler thought it possible to explain the diversity of the observed phenomena (including electricity, light, gravitation, and even the principle of least action) by the hypothetical mechanical properties of ether. He also had to introduce magnetic whirls into the doctrine of magnetism; these are even thinner and move more quickly than ether.

In physics Euler built up many artificial models and hypotheses which were short-lived. But his main concept of the unity of the forces of nature acting deterministically in some medium proved to be important for the development of physics, owing especially to *Lettres à une princesse d'Allemagne*. Thus, his views on the nature of electricity were the prototype of the theory of electric and magnetic fields of Faraday and Maxwell. His theory of ether influenced Riemann.

Euler's works on optics were widely known and important in the physics of the eighteenth century. Rejecting the dominant corpuscular theory of light, he constructed his own theory in which he attributed the cause of light to peculiar oscillations of ether. His *Nova theoria lucis et colorum*(1746)[38]explained some, but not all, phenomena. Proceeding from certain analogies that later proved incorrect, Euler concluded that the elimination of chromatic aberration of optic lenses was possible (1747); he conducted experiments with lenses filled with water to confirm the conclusion. This provoked objections by the English optician Dollond, who, following Newton, held that dispersion was inevitable. The result of this polemic, in which both parties were partly right and partly wrong, was the creation by Dollond of achromatic telescopes (1757), a turning point in optical technology. For his part, Euler, in his *Dioptrica*, laid the foundations of the calculation of optical systems.

# NOTES

All works cited are listed in the BIBLIOGRAPHY. References to Euler's *Opera omnia*(see [1] in BIBLIOGRAPHY) include series and volume number.

1. 20, p. 75

2. "Constructio linearum isochronarum in medio quocunque resistente," in 1, 2nd ser., VI, p. 1.

3. "Methodus inveniendi traiectorias reciprocas algebraicas," in 1, 1st ser., XXVII, p. 1.

4. To be published in 1, 2nd ser., XX.

5. 1, 3rd ser., 1p. 181.

6. 26, p. 51.

7. 13, **II** , p. 182.

8*Einleitung zur Rechen-Kunst zum Gebrauch des Gymnasii bey der Kayserlichen Academie der Wissenschafften in St. Petersburg* (St. Petersburg, 1738-1740). See 1, 3rd ser., **II,** 1-303.

9.*Mechanica sine motus scientia analytice exposita*, 2 vols. (St. Petersburg, 1736). See 1, 2nd ser., I and II.

10. 20, p. 77.

11.*Neue Grudsätze der Artillerie aus dem Englischen des Herrn Benjamin Robins übersetzt und mit vielen Anmerkungen versehen* (Berlin, 1745). See 1, 2nd ser., XIV.

12. See 1, 2nd ser., XIV, 468-477.

13.*Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes* (Lausanne-Geneva, 1744). See 1, 1st ser., XXIV.

14.*Theoria motuum planetarum et cometarum* (Berlin, 1744). See 1, 2nd ser., XXVIII, 105-251.

15.*Introductio in analysin infinitorum, 2 vols*. (Lausanne, 1748). See 1, 1st ser., VIII and IX.

16.*Scientia navalis,* 2 vols. (St. Petersburg, 1749). See 1, 2nd ser., XVIII and XIX.

17.*Theoria motus lunae*(Berlin, 1753). See 1, 2nd ser., XXIII, 64-336.

18.*Institutiones calculi differentialis cum eius usu in analyst finitorum ac doctrina serierum*(Berlin, 1755). See 1, 1st ser., **X** .

19.*Theoria motus corporum solidorum seu rigidorum ex primis nostrae cognitionis principiis stabilita*... (Rostock-Greifswald, 1765). See 1, 2nd ser., III and IV.

20. The work, which comprises 234 letters, was published at St. Petersburg in 3 vols. The first two vols. (letters 1-154) appeared in 1768; vol. **III** appeared in 1772. See 1, 3rd ser., XI and XII.

21.*Gedancken von den Elementen der Cörper, in welchen das Lehr-Gebäude von den einfachen Dingen und Monaden geprüfet und das wahre Wesen der Cörper entdecket wird* (Berlin, 1746). See 1, 3rd ser., **II,** 347-366.

22. "Exposé concernant l'examen de la lettre de M. de Leibnitz, alléegueé par M. le Professeur Koenig, dans le mois de mars 1751 des Actes de Leipzig, à l'occasion du principe de la moindre action." See 1, 2nd ser., V, 64-73.

23. The work was first published at St. Petersburg in Russian (vol. I, 1768; vol. II, 1769). It then appeared in a two-volume German edition (St. Petersburg, 1770). See 1, 1st ser., I.

24.*Theoria motuum lunae, nova methodo pertractata (St. Petersburg, 1772). See 1, 2nd ser., XXII.*

25. The work was published sequentially, in 3 vols., at St. Petersburg. Vol. I deals with principles of optics (1769); vol. II with construction of telescopes (1770); and vol. III with construction of microscopes (1771). See 1, 3rd ser., III and IV.

26. The work's 3 vols. were published sequentially in St. Petersburg in 1768, 1769, and 1770. See 1, 1st ser., III and XIII.

27. To be published in 1, 2nd ser., XXI.

28. See 1, 3rd ser., VII, 200-247.

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30. See 17.

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32. See 50, chs.1-2.

33. "Recherches sur la précession des équinoxes et sur la nutation de l'axe de la terre." See 1, 2nd ser., XXIX, 92-123.

34. See 1, 2nd ser., XII, 2-132.

35. See 1, 2nd ser., XII, 133-168.

36. See 1, 2nd ser., XII, 92, for the original French.

37. See 1, 2nd ser., XV, pt. 1, 1-39, 80-104, 157-218.

38. See 1, 3rd ser., V, 1-45.

# BIBLIOGRAPHY

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Euler's enormous correspondence (approximately 300 addressees), which he conducted from 1726 until his death, has been only partly published. For an almost complete description, with summaries and indexes, see (37) below. For his correspondence with Johann I Bernoulli, see (2) and (3); with Nikolaus I Bernoulli (2) and (4); with Daniel Bernoulli (2) and (3); with C. Goldbach (2) and (5); with J.-N. Delisle (6); with Clairaut (7); with d'Alembert (3) and (8); with T. Mayer (9); with Lagrange (10); with J.H. Lambert (11); with M. V. Lomonosov (12); with G. F. Müller (13); with J. D. Schumacher (13); with King Stanislas II (14); and with various others (15).

Euler's complete works are in the course of publication in a collection that has been destined from the outset to become one of the monuments of modern scholarship in the historiography of science: *Leonhardi Euleri Opera omnia* (Berlin-Göttingen-Leipzig-Heidelberg, 1911-). The *Opera omnia* is limited for the most part to republishing works that Euler himself prepared for the press. All texts appear in the original language of publication. Each volume is edited by a modern expert in the science it concerns, and many of the introductions constitute full histories of the relevant branch of science in the seventeenth and eighteenth centuries. Several volumes are in course of preparation. The work is organized in three series. The first series (*Opera mathematica*) comprises 29 vols. and is complete. The second series (*Opera mechanica et astronomica*) is to comprise 31 vols. and still lacks vols. XVI, XVII, XIX, XX, XXI, XXIV, XXVI, XXVII, and XXXI. The third series (*Opera physica, Miscellanea, Epistolae*) is to comprise 12 vols. and still lacks vols. IX and X. Euler's correspondence is not included in this edition.

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