ARITHMETICA LOGARITHMICA

§ 1.1 *Synopsis: Chapter One.*

The first chapter establishes in a general way how logarithms may be defined. By means of tabulated numerical examples (Table 1-1), Briggs shows how a set of positive numbers in continued proportion *P, i.e.* a geometric progression, can have associated with it a variety of arithmetic progressions of positive terms *L*, each element of *L* is associated with an element of *P* in a 1:1 correspondence: Briggs uses the word *adjoined* to describe this correspondence. For explanatory purposes here, we consider *P* to be an ordered finite set of N elements: $\{p_1, p_2, p_3, \ldots, p_N\}$, with a typical member of the set defined by $p_j = r^j$, for some integer $r > 0$ in the first chapter. The corresponding set of numbers in the set *L*: $\{l_1, l_2, l_3, \ldots, l_N\}$, may be conveniently chosen. The elements of *P* are said by Briggs to be in *continued or successive proportion,* while the corresponding members of *L* are called logarithms, the term coined by John Napier. These logarithms, or 'ratio numbers', at this early stage of the development, can be related to the index *j* in numerous ways, the simplest being *j* itself: a measure of how many times *r* has been multiplied by itself, or the 'power' of the number. Thus, the index is a number which lets you point at another number, as it were, though the notation of setting the index as a post superscript was yet to come. Also, the logarithm will continue to be defined if *j* is not a whole number, as Briggs sets out in the following chapters.

Initially then, Briggs defines logarithms in an introductory manner according to the above ideas: for any two consecutive elements in *P*, the quotient p_{j+1}/p_j is formed, to which there corresponds a constant difference of logarithms $d = l_{j+1} - l_j$, from some particular set *L* of logarithms chosen, as shown in Table 1-1. The idea of subtracting indices, (or of numbers related to the indices) for a quotient of powers, is thus fundamental to this definition of logarithms. From Ch. 6 onwards, natural logarithms of primes are worked out and changed to base 10.

<u>Lemma 1</u>: The difference between any two logarithms in the table is proportional to the interval between their indices, which difference itself is a multiple of *d.* In Table 1-1, the index is the number of times 2 multiplies itself, and corresponds to the rows numbered from 1 to 8 sequentially. Now, for appropriate values of *i, m*, and *n*, consider an interval to be the difference *m* corresponding to rows *i* and $i + m$; similarly, rows *i* and $i + n$ give a second interval with a difference *n*. If the logarithms l_i and l_{i+m} are given for the first interval, then the unknown logarithm l_{i+n} can be found from proportion, as *n* is known. For $(l_{i+m} - l_i)/m = (l_{i+n} - l_i)/n = d;$ in Table 1-1, $d = 1$ for columns A and B; for column C, $d = 3$; and for column D, $d = -3$.

<u>Lemma 2</u>: If *p, q, r,* and *s* are any four numbers such that $p - q = r - s$ then $p + s = q + r$. As Briggs shows in A2 of the next chapter, he has in mind the ratio of the numbers $a/b = c/d$, where *p, q, r*, and *s* are the respective logarithms of the positive numbers *a, b, c*, and *d.* Hence, the sum of the logarithms is the same in the products *ac* and *bd*.

§1.2

Chapter One.

Concerning the definition of logarithms and the etymology of the name.

Logarithms are numbers which, adjoined to numbers in proportion, maintain equal differences[1].

For any given numbers whatsoever, the other numbers or logarithms, which are different from these first ones, and which conveniently agree with the general definition of logarithms, can be added on, and they do offer some appreciable advantage [in performing calculations]. As an example, if the numbers in continued proportion are[2]: 1, 2, 4, 8, 16, 32, 64, 128,… then the adjoined numbers designated by *A, B, C*, or *D* or others, can act as logarithms for these, as you see here [Table 1-1]. But a single kind of logarithm is to serve from this table, in order that the differences will be equal, with the one set of logarithms either increasing or decreasing, as often as the numbers with which they are adjoined are in proportion; so that conveniently, logarithms may be called*: the equally differing companions of proportional numbers.* Hence, the adjoined numbers seem to have been called logarithms [*lit.* ratio number] by their most distinguished inventor [John Napier], because they present numbers to us that maintain a constant ratio between themselves[3].

	A	B	C	D
1	1	5	5	35
2	2	6	8	32
4	3	7	11	29
8	4	8	14	26
16	5	9	17	23
32	6	10	20	20
64	7	11	23	17
128	8	12	26	14
numbers in proportion	Log	Log	Log	Log

[Table 1-1.]

Furthermore, so that we can reach a better understanding of the purposes of these logarithms to be produced, certain Lemmas are considered.

First Lemma

If any numbers whatever are established increasing or decreasing equally, the differences of these are in proportion with the intervals of the same[4].

For the first, third, and eighth numbers 35, 29, 14 can be selected from the numbers designated *D* [in Table 1-1]; between the first and third there are 2 intervals; between the third and eighth there are 5 intervals. I assert that the first to third difference 6, is to the difference of the third and eighth 15, as two is to five.

Therefore in a series of numbers in continued proportion, from the logarithms of any two numbers given, we can find the logarithm of any other number whatever in the series.
Indeed, both the intervals between these numbers themselves and the third number are given, also given is the difference of the logarithms : therefore, for two given intervals and the difference of the logarithms for the first interval, with the other interval given, the difference of the logarithms required for the second interval is found from proportion. As an example: let 4, 6, 9, $13^{1}/_{2}$, $20^{1}/_{4}$, $30^{3}/_{8}$, be the numbers in continued proportion, and the logarithms of the first and third given numbers are 060206 and 095424, and 035218 is the difference of these. The logarithm of the number $30^{3}/_{8}$ is required, namely the sixth of the given numbers. The given interval between the first and the third of the numbers is two, between the third and the sixth three; the given difference of the logarithms is 035218 ; the fourth proportional number sought 052827 is the difference of the logarithms sought, which added to the logarithm of the third number gives 148251, the logarithm of the sixth number required[5]. As this table [1-2] shows:

continued proportion	log	Logarithm		
4	7	060206		
6			difference given	035218
9	11	095424		
$13^{1}/_{2}$				
$20^{1}/_{4}$			difference sought	052827
$30^{3}/_{8}$	17	148251		

proportions		
	2----	given intervals
	3----	
	035218	given difference
	052827	sought difference

[Table 1-2]

Second Lemma

If from four numbers, the first exceeds the second as much as the third exceeds the fourth: then the sum of the first and forth is equal to the sum of the second and third, and conversely. As, for example the numbers 9, 5, 15, 11: so the sum of the means as of the extremes is 20. See Proposition 4, Book 1, of the deductions in Bachet's *Diophantus*, [the 1621 edition].

And these two lemmas illustrate well enough [our] particular feelings for [the use of] a kind of logarithms.

§1.3 *Notes On Chapter One.*

[1] According to Briggs' Table 1-1, we have 1, 2, 4, 8 as four numbers in proportion, and we may take the corresponding numbers in column C as their logarithms, by way of example: 5, 8, 11, 14. The ratio of 2 to 1 is the same as the ratio of 8 to 4, or 2/1 = 8/4. With the first ratio we associate the difference of the logarithms 8 - 5 or 3, while with the second ratio we associate the difference of the logarithms 14 - 3 or 3. The same above rule is found for any numbers in the table, or for any example we may construct, and for any numbers we choose as logarithms.

N.B. By continued proportions, the original writer had in mind a sequence of positive numbers *a, b, c, d,...* that satisfies a relation that we would now express as: $a/b = b/c = c/d$ ...

[2] Briggs used periods to delineate a series of numbers in continued proportion, such as 1.2.4.8.16.32.64.128,…In order to increase the readability of the text, we have changed the periods to commas.

[3] Logarithms are seen to be multiples of some common difference, as one would expect for an A.P. The word 'logarithm' is derived from two Greek words, 'logos' here meaning 'ratio' or perhaps 'reckoning' and 'arithmos', meaning 'number. An article in the *Mathematical Gazette* for 1934, pp. 92 - 205, *John Napier*, by W. R. Thomas, explores the possible origins of the word, and also sheds some light on the life of John Napier. The Introductory Chapter of this work may also be consulted.

[4] At this time there were hardly any means of describing mathematical operations on numbers, except by labourously writing out instructions. In this first lemma, a set of numbers in continued proportion may be considered to occupy the successive rows of the first column of a table: for each

of these numbers there is an adjoining logarithm, in this case initially taken from the arbitrary A.P. in column D as an easy example, and subsequently using the actual logarithms from Briggs' tables of logarithms (that formed the bulk of the latter part of the *Arithmetica*, see Table 1-2, where another arbitrary A.P. is given as a guide). Briggs takes the actual row numbers as indices from which the interval between two terms can be evaluated, in an obvious way: each application of the proportionality moves to the number placed in the next row of the first column, to which there corresponds an equal increase or decrease in the logarithm in column 2 or 3 in Table 1-2. Thus, the row index can be associated with either the numbers in continued proportion, or their logarithms: the latter in this case.

5 Consider the first set of numbers: 7, 9,11,13,15,17,… to be the logs of the given continued proportions. In this case, Briggs' instructions results in: 'log' $30^3/_8$ = 'log' 9 + ('log' 9 - 'log' 4)/2 $\times$ 3= (11 - 7)/2 $\times$ 3 = 17; where 'log' is used here to avoid confusion with the base10 logarithm of a number. Note: for the actual base 10 logarithms Briggs uses no decimal point - we would normally write log 4 = 0.60206, etc. An initial zero indicates the characteristic (to be discussed in Ch.3) is zero. A similar calculation can be performed with the logarithms from the tables in column 3 of Table 1-2. We must bear in mind that the *Arithmetica* is an expository work, and the nature of logarithms is to be gradually introduced to the reader. The idea of associating an arbitrary A.P. with the numbers in a G.P. is replaced by a more sophisticated method, whereby essentially the natural logarithms of numbers are found, and changed to base 10 in an intuitive manner.

ARITHMETICA
LOGARITHMICA

§1.4. Caput Primus. [p.1.]

De Logarithmorum definitione & nominis notatione.

Logarithmi sunt numeri qui proportionalibus adiuncti aequales servant differentias.

Datis quibuscunque numeris alij numeri ijque diversi ijs adjungi poterunt qui non incommode generali Logarithmorum definitioni conveniant & usum aliquem non ingratum praebere poterunt. ut si numeri continue proportionales fuerint 1.2.4.8.16.32.64.128. poterunt ijs pro Logarithmis adjungi numeri insigniti *A*, vel *B*, vel *C*, vel *D,* ut hic vides. vel alij, modo hoc unicum servetur, ut Logarithmorum, una crescentium vel decrescentium differentiae sint aequales, quoties numeri quibus sunt adjuncti sunt proportionales. ut non incommode Logarithmi dici possint, *numerorum proportionalium comites aequidifferentes.* Qui ideo videntur a clarissimo Inventore Logarithmi nominati, quia numeros nobis exhibent eandem inter se servantes rationem.

	A	B	C	D
1	1	5	5	35
2	2	6	8	32
4	3	7	11	29
8	4	8	14	26
16	5	9	17	23
32	6	10	20	20
64	7	11	23	17
128	8	12	26	14
numeri proportion-ales.	Logar.	Logar.	Logar.	Logar.

Atque ut horum Logarithmorum naturum & affectiones melius noscamus, Lemmata quaedam sunt considerenda.

Lemma primum.

Si quotlibet numeri statuantur aequaliter crescentes vel decrescentes eorum differentiae sunt intervallis eorundem proportionales.

Ut in numeris *D* signatis sumantur primus. tertius, & octavus, 35.29.14 inter primum & tertium sunt duo intervalla, inter tertium & octavum sunt quinque intervalla. aio differentiam primi & tertij 6, esse ad differentiam tertij & octavi 15, ut duo ad quinque.

Idcirco in serie numerorum continue proportionium, datis duorum quorumcunque Logarithmis; poterimus alterius cuiusvis Logarithmum invenire. Datur enim horum numerorum inter se intervallum & tertij ab
.2. horum alterutro, datur etiam datorum Logarithmorum differentia. Datis ergo duobus intervallis & Logarithmorum datorum differentia, altera dati & quaesiti Logarithmorum differentia proportionalis inventa erit.
ut sunto $4.6.9.13^{1}/_{2}.20^{1}/_{4}.30^{3}/_{8}$ continue proportionales, & sint Logarithmi primi & tertij numeri dati, & 060206, & 095424, eorum differentia 035218. quaeritur Logarithmus numeri $30^{3}/_{8}$, sexti nempe datorum. data intervalla inter primum numerum & tertium sunt duo, inter tertium & sextum tria; data Logarithmorum differentia 035218, quartus quaesitus numerus proportionalis 0528273 est differentia Logarithmorum quaesita, quae tertij numeri dato Logarithmo adiecta, dat 148251 Logarithmum sexti numeri quaesitum. ut hic vides.

4	7	060206	
6			diff. data 035218
9	11	095424	
$13^{1}/_{2}$			
$20^{1}/_{4}$			diff. quart. 052827
$30^{3}/_{8}$	17	148251	
cont. prop.	log.	Logarith.	

prop. { 2---- } 3---- } intervalla data
035218 data } differentia
052827 quaesita }

Lemma secundum.

[p.2.]

Si e quatuor numeris, quantum primus superat secundum. tantundem tertius superet quartum: erit summa primi & quarti, aequalis summae secundi & tertij: & contra. vt 9.5.15.11 summa tam mediorum quam extremorum est 20. vide 4. prop.lib.1 porismatum Bacheti in Diaphantum.

Atque haec duo Lemmata, Logarithmorum in genere, praecipuas affectiones satis illustrant.