MONOIDS $\mathrm{Mon}\langle a, b : a^\alpha b^\beta a^\gamma b^\delta = b\rangle$ ADMIT FINITE COMPLETE REWRITING SYSTEMS

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Abstract. We prove that every monoid $\mathrm{Mon}\langle a, b : a^\alpha b^\beta a^\gamma b^\delta = b\rangle$ admits a finite complete rewriting system. Furthermore we prove that $\mathrm{Mon}\langle a, b : ab^2a^2b^2 = b\rangle$ is non-hopfian, providing an example of a finitely presented non-residually finite monoid with linear Dehn function.

1. Introduction

The solubility of the word problem for one-relator monoids is a long-standing open question. In a series of papers by Sergei Adian and his students it was proved that the word problem for one-relator monoids can be reduced to the cases $\mathrm{Mon}\langle a, b : aUb = bVb\rangle$ and $\mathrm{Mon}\langle a, b : aUb = b\rangle$; we refer the reader to the very nice survey [1] and references therein. The methods of Adian's school is mostly combinatorics on words, and sometimes the proofs using these methods can become quite technically involved. On the other hand, Louxin Zhang showed in [7] how powerful the tools of rewriting systems can be in trying to prove that the word problem for one-relator semigroups is decidable. A remarkable paper of Yuji Kobayashi [6] showed that every one-relator monoid satisfies the condition FDT, and since every monoid presented by a finite complete rewriting system satisfies FDT, it prompted Kobayashi to ask:

Open Problem 1.1. Does every one-relator monoid admit a finite complete rewriting system?

The aim of this note is to show that monoids $\mathrm{Mon}\langle a, b : a^\alpha b^\beta a^\gamma b^\delta = b\rangle$ admit finite complete systems, see Section 3. Notice that these monoids fall within one of the two important classes identified by Adian's school. After that, in Section 4, we will prove that $\mathrm{Mon}\langle a, b : ab^2a^2b^2 = b\rangle$ is non-hopfian. This gives an example of a non-residually finite finitely presented monoid with linear Dehn function. This is significant because the analogous question for finitely presented groups with linear Dehn function, which are of course the hyperbolic groups, is an important open problem. Finally, in Section 5 we will state our feelings about general monoids $\mathrm{Mon}\langle a, b : aUb = b\rangle$ and pose some questions.

2. Preliminaries

By a *rewriting system* (A, R) we mean a finite alphabet A and a subset $R \subseteq A^* \times A^*$, where A^* stands for the free monoid over A. Every pair (l, r) from R is called a *rule* and normally is written as $l \to r$. For $x, y \in A^*$ we write $x \to y$, if

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there exist $\alpha, \beta \in A^*$ and a rule $l \to r$ from R such that $x = \alpha l \beta$ and $y = \alpha r \beta$. Denote by $\to^*$ the transitive reflexive closure of $\to$. A rewriting system (A, R) is called

- *confluent* if for every words $w, x, y \in A^*$ such that $w \to^* x$ and $w \to^* y$, there exists $W \in A^*$ such that $x \to^* W$ and $y \to^* W$;
- *terminating* if there is no infinite derivation $x_0 \to x_1 \to x_2 \to \cdots$.

Confluent terminating rewriting systems, which are also called *complete systems*, give a very convenient way of working with finitely generated monoids. For, if a monoid is presented by $M = \mathrm{Mon}\langle A : l_i = r_i \quad i \in I\rangle$ and it turns that $S = (A, \{l_i \to r_i\}_{i \in I})$ is complete, then the elements of M are in bijection with the *normal forms* for S, i.e. those words from A^* which do not include any subword l_i, and to find the normal form for a word $w \in A^*$, we just need to apply the relation $\to$ successively to w as many times as we can (this process must stop by the termination condition) and the result will always be the same word depending only on the element of M that w represents.

We refer the reader to the monograph of Ronald Book and Friedrich Otto [3] for more background information on rewriting systems.

Let us provide our two final definitions. Let $\mathrm{Mon}\langle A : R\rangle$ be a finite presentation for a monoid M. For two words $x, y \in A^*$, equal in M, denote by

- $d(x, y)$ the minimal number of relations from R that need to be applied to obtain x from y.
- $s(x, y)$ the least possible value of $\sup\{|w_i| : 0 \le i \le k\}$ for all derivations $x = w_0 \sim w_1 \sim \cdots \sim w_k = y$, where $p \sim q$ stands for applying a single relation from R.

Then

$$\mathbf{d}_n(M) = \sup\{d(x, y) : x, y \in A^*,\ x =_M y,\ |x|_A, |y|_A \le n\}$$

is called the *Dehn function* of M, and

$$\mathbf{sp}_n(M) = \sup\{s(x, y) : x, y \in A^*,\ x =_M y,\ |x|_A, |y|_A \le n\}$$

is called the *space function* of M.

3. Finite Complete Systems

Theorem 3.1. *Every monoid $M = \mathrm{Mon}\langle a, b : a^\alpha b^\beta a^\gamma b^\delta = b\rangle$ admits a finite complete system.*

Proof. If there are no overlaps of the word $a^\alpha b^\beta a^\gamma b^\delta$ with itself, then

$$a^\alpha b^\beta a^\gamma b^\delta \quad \to \quad b$$

is a complete rewriting system for M. The word $a^\alpha b^\beta a^\gamma b^\delta$ only overlaps with itself when $\beta \ge \delta$ and $\gamma \ge \alpha$. Thus we may assume that $a^\alpha b^\beta a^\gamma b^\delta \equiv a^p b^{q+s} a^{r+pk} b^s$ where $p, s, k \ge 1$, $q \ge 0$ and $0 \le r < p$.

Case 1: $s = 1$

Overlapping $a^p b^{q+1} a^{r+pk} b \to b$ with itself, we obtain a new rule $a^p b^{q+1} a^{r+p(k-1)} b \to b^{q+1} a^{r+pk} b$. Then successively overlapping the newly obtained rules with the initial one, we obtain the following finite complete system for M:

$$\begin{aligned} a^p b^{q+1} a^{r+pk} b &\quad \to \quad b \\ a^p b^{q+1} a^{r+pi} b &\quad \to \quad b^{q+1} a^{r+p(i+1)} b, \quad 0 \le i \le k-1. \end{aligned}$$

Case 2: $s > 1$ and $r > 0$

By the same tactics as in **Case 1**, we obtain the following finite complete system for M:

$$\begin{array}{rcl} a^p b^{q+s} a^{r+pk} b^s & \to & b \\ a^p b^{q+s} a^{r+pi} b & \to & b^{q+1}(a^{r+pk} b^{q+2s-1})^{k-1-i} a^{r+pk} b^s, \quad 0 \le i \le k-1. \end{array}$$

Case 3: $s > 1$, $r = 0$ and $k = 1$

It is easy to see that M admits the following finite complete system:

$$\begin{array}{rcl} a^p b^s & \to & x \\ x b^q x & \to & b \\ x b^{q+1} & \to & b^{q+1} x. \end{array}$$

Case 4: $s > 1$, $r = 0$ and $k \ge 2$

We have the relation $a^p b^{q+s} a^{pk} b^s = b$. We add a new letter $x = a^{pk} b^s$ and then $a^p b^{q+s} x = b$.

Now, $a^{p(k-1)} b = a^{pk} b^{q+s} x = x b^q x$, and so $\underline{a^p x b^q x b^{s-1} = x}$. Since $a^p b \cdot b^{q+s-1} x = b$, we have that $\underline{a^p b} = a^{p(k-1)} b \cdot (b^{q+s-1} x)^{k-2} = \underline{x b^q x (b^{q+s-1} x)^{k-2}}$. Then

$$\underline{b} = a^p b^{q+s} x = x b^q x (b^{q+s-1} x)^{k-2} \cdot b^{q+s-1} x = \underline{x b^q x (b^{q+s-1} x)^{k-1}}.$$

This yields

$$\begin{array}{rcl} \underline{x b^q x (b^{q+s-1} x)^{k-2} b^{q+s}} & = & x b^q x (b^{q+s-1} x)^{k-2} b^{q+s-1} \cdot x b^q x (b^{q+s-1} x)^{k-1} \\ & = & x b^q x (b^{q+s-1} x)^{k-1} \cdot b^q x (b^{q+s-1} x)^{k-1} \\ & = & \underline{b^{q+1} x (b^{q+s-1} x)^{k-1}}. \end{array}$$

The underlined relations give us the following rewriting system, defining M:

$$\begin{array}{rcl} a^p x b^q x b^{s-1} & \to & x \\ a^p b & \to & x b^q x (b^{q+s-1} x)^{k-2} \\ x b^q x (b^{q+s-1} x)^{k-1} & \to & b \\ x b^q x (b^{q+s-1} x)^{k-2} b^{q+s} & \to & b^{q+1} x (b^{q+s-1} x)^{k-1}. \end{array}$$

If $q < s-1$, one readily checks that this system is confluent and terminating (regardless whether $k > 2$ or $k = 2$).

If $q \ge s-1$, then

$$a^p x b^{q+1} = a^p x b^q \cdot x b^q x (b^{q+s-1} x)^{k-1} = x b^{q-(s-1)} x (b^{q+s-1} x)^{k-1},$$

and adding the rule

$$a^p x b^{q+1} \to x b^{q-(s-1)} x (b^{q+s-1} x)^{k-1}$$

to the system, we obtain the required finite complete system. □

4. Non-Hopfian Example

Example 1. The monoid $M = \mathrm{Mon}\langle a, b : ab^2a^2b^2 = b\rangle$ is non-hopfian.

Proof. Our example falls within Case 4 of the proof of Theorem 3.1. By letting $x = a^2b^2$, we obtain the following complete system for M:

$$\begin{aligned} ax^2b &\to x \\ ab &\to x^2 \\ x^2bx &\to b \\ x^2b^2 &\to bxbx. \end{aligned}$$

Consider the assignment $a \mapsto a$ and $b \mapsto bab$. Since

$$\begin{aligned} a(bab)^2a^2(bab)^2 &\to x^2 \cdot x^2bx^2 \cdot ax^2 \cdot x^2bx^2 \\ &\to x^2bx \cdot ax^2b \cdot x \\ &\to bx^2 \end{aligned}$$

and $bab \to bx^2$, we have that the assignment lifts to a homomorphism. Under this homomorphism ab^2 maps to

$$abab^2ab \to x^2 \cdot x^2bx^2 \to x^2bx \to b,$$

and so the homomorphism is surjective. If this homomorphism were bijective, then we would have that the inverse of this homomorphism would be a homomorphism given by $a \mapsto a$ and $b \mapsto ab^2$. But under this assignment the relation $ab^2a^2b^2 = b$ does not hold, for:

$$\begin{aligned} a \cdot ab^2ab^2a^2 \cdot ab^2ab^2 &= a^2b^2 \cdot ab^2 \cdot a \cdot a^2b^2 \cdot ab^2 \\ &= xab^2axab^2 \\ &\to x^3bax^3b, \end{aligned}$$

which does not reduce to $ab^2 = x^2b$. Thus M is non-hopfian. □

Remark 4.1. Malcev's Theorem asserts that every finitely presented residually finite semigroup is hopfian. Thus the monoid M from Example 1 is non-residually finite. It also follows immediately from the complete system for M that M has linear Dehn function.

On the other hand, for groups, it is still an important open question whether every hyperbolic group is residually finite. (Finitely presented groups with linear Dehn function are hyperbolic; see [4].)

5. Remarks and Questions

We have proved that every monoid $\mathrm{Mon}\langle a, b : a^\alpha b^\beta a^\gamma b^\delta a^\varepsilon b^\varphi = b\rangle$ admits a finite complete system and will shortly make the proof available as a preprint. The proof of this result, in comparison to that of Theorem 3.1, is already very technical and gives little hope that it is possible to prove that every monoid $\mathrm{Mon}\langle a, b : aUb = b\rangle$ admits a finite complete system just by straightforward method. Yet, analysing the cases appearing in that proof, and looking at the proof of Theorem 3.1, we noticed that the one-relator monoids under consideration have at most quadratic Dehn functions and linear space functions. This prompts us to raise

Open Problem 5.1. Is it true that

(1) every monoid $\mathrm{Mon}\langle a, b : aUb = b\rangle$ has at most quadratic Dehn function?
(2) every monoid $\mathrm{Mon}\langle a, b : aUb = b\rangle$ has linear space function?

The reader may wish to consult a brilliant paper of Victor Guba [5] on some other possible approaches how to deal with monoids $\mathrm{Mon}\langle a, b : aUb = b\rangle$.

Another question we were trying to settle is whether every monoid $\mathrm{Mon}\langle a, b : a^\alpha b^\beta a^\gamma b^\delta = b\rangle$ admits a *length-non-increasing* finite complete system (that is, where the rewriting rules $l \to r$ are all such that $|l| \geq |r|$). Using Knuth–Bendix completion in GAP, we have thus far eliminated all our suspected counterexamples, so we simply ask the general question:

Question 5.2. Do all monoids $\mathrm{Mon}\langle a, b : a^\alpha b^\beta a^\gamma b^\delta = b\rangle$ admit length-non-increasing finite complete systems?

Note that it follows from the results of Günther Bauer and Friedrich Otto [2] that there do exist monoids admitting finite complete systems but not admitting finite complete system which do not increase the lengths.

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