

Chebyshev, Pafnuty Lvovich | Encyclopedia.com

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(b. Okatovo, Kaluga region, Russia, 16 May 1821; d. [St. Petersburg](#), Russia, 8 December 1894),

mathematics.

Chebyshev's family belonged to the gentry. He was born on a small estate of his parents, Lev Pavlovich Chebyshev, a retired army officer who had participated in the war against Napoleon, and Agrafena Ivanovna Pozniakova Chebysheva. There were nine children, of whom, besides Pafnuty, his younger brother, Vladimir Lvovich, a general and professor at the Petersburg Artillery Academy, was also well known. Vladimir paid part of the cost of publishing the first collection of Pafnuty's works (1, la).

In 1832 the Chebyshevs moved to Moscow, where Pafnuty completed his secondary education at home. He was taught mathematics by P. N. Pogorelski, one of the best tutors in Moscow and author of popular textbooks in elementary mathematics.

In 1837 Chebyshev enrolled in the department of physics and mathematics (then the second section of the department of philosophy) of Moscow University. Mathematical disciplines were then taught brilliantly by N. D. Brashman and N. E. Zernov. Brashman, who always directed his pupils toward the most essential problems of science and technology (such as the theory of integration of algebraic functions or the calculus of probability, as well as recent inventions in [mechanical engineering](#) and hydraulics), was especially important to Chebyshev's scientific development. Chebyshev always expressed great respect for and gratitude to him. In a letter to Brashman, discussing expansion of functions into a series by means of continued fractions, Chebyshev said: "What I said illustrates quite sufficiently how interesting is the topic toward which you directed me in your lectures and your always precious personal talks with me" (2, II, 415). The letter was read publicly at a meeting of the Moscow Mathematical Society on 30 September 1865 and printed in the first issue of *Matematicheskyy sbornik* ("Mathematical Collection"), published by the society in 1866. Chebyshev was one of the first members of the society (of which Brashman was the principal founder and the first president).

As a student Chebyshev wrote a paper, "Vychislenie korney uravneny" ("Calculation of the Roots of Equations"), in which he suggested an original iteration method for the approximate calculation of real roots of equations $y = f(x) = 0$ founded on the expansion into a series of an inverse function $x = F(y)$. The first terms of Chebyshev's general formula are

where α is an approximate value of the root x of the equation $f(x) = 0$ differing from the exact value by sufficiently little. Choosing a certain number of terms of the formula and successively calculating from the chosen value a further approximations $\alpha_1, \alpha_2, \dots$, it is possible to obtain iterations of different orders. Iteration of the first order is congruent with the widely known Newton-Raphson method: $\alpha_{n+1} = \alpha_n - f(\alpha_n)/f'(\alpha_n)$. Chebyshev gives an estimation of error for his formula. This paper, written by Chebyshev for a competition on the subject announced by the department of physics and mathematics for the year 1840–1841, was awarded a silver medal, although it undoubtedly deserved a gold one. It was published only recently (2, V).

In the spring of 1841 Chebyshev graduated from Moscow University with a candidate (bachelor) of mathematics degree. Proceeding with his scientific work under Brashman's supervision, he passed his master's examinations in 1843, simultaneously publishing an article on the theory of multiple integrals in Liouville's *Journal des mathématiques pures et appliquées* and in 1844 an article on the convergence of Taylor series in Crelle's *Journal für die reine and angewandte Mathematik* (see 1a, I; 2, II). Shortly afterward he submitted as his master's thesis, *Opyt elementarnogo analiza teorii veroyatnostey* ("An Essay on an Elementary Analysis of the Theory of Probability"; (see 2, V). The thesis was defended in the summer of 1846 and was accompanied by "démonstration élémentaire d'une proposition générale de la théorie des probabilités" (*Journal für die reine and angewandte Mathematik*, 1846; see 1a, I; 2, II), which was devoted to Poisson's law of large numbers. These works aimed at a strict but elementary deduction of the principal propositions of the theory of probability; of the wealth of mathematical analysis Chebyshev used only the expansion of $\ln(1+x)$ into a power series. In the article on Poisson's theorem we find an estimation of the number of tests by which it is possible to guarantee a definite proximity to unit probability of the assumption that the frequency of an event differs from the arithmetic mean of its probabilities solely within the given limits. Thus, even in Chebyshev's earliest publications one of the peculiar aspects of his work is manifest: he aspires to establish by the simplest means the most precise numerical evaluations of the limits within which the examined value lies.

It was almost impossible to find an appropriate teaching job in Moscow, so Chebyshev willingly accepted the offer of an assistant professorship at Petersburg University. As a thesis *pro venia legendi* he submitted “Ob integririvanii pomoshchyu logarifimov” (“On Integririvan by Means of Logarithms”), written, at least in the first draft, as early as the end of 1843. The thesis, defended in the spring of 1847, incidentally solved a problem of integration of algebraic irrational functions in the final form that had been posed shortly before by Ostrogradski. The thesis was published posthumously, as late as **1930** (see 2, V), but Chebyshev included its principal results in his first publication on the subject in 1853.

In September 1847, at Petersburg University, Chebyshev began lecturing on higher algebra and the theory of numbers. Later he lectured on numerous other subjects, including [integral calculus](#), elliptic functions, and calculus of finite differences; but he taught the theory of numbers as long as he was at the university (until, 1882). From 1860 he regularly lectured on the theory of probability, which had previously been taught for a long time by V. Y. Bunyakovski. A. M. Lyapunov, who attended Chebyshev’s lectures in the late 1870’s, thus characterized them:

His courses were not voluminous, and he did not consider the quantity of knowledge delivered; rather, he aspired to elucidate some of the most important aspects of the problems he spoke on. These were lively, absorbing lectures; curious remarks on the significance and importance of certain problems and scientific methods were always abundant. Sometimes he made a remark in passing, in connection with some concrete case they had considered, but those who attended always kept it in mind. Consequently, his lectures were highly stimulating; students received something new and essential at each lecture; he taught broader views and unusual standpoints [4, p. 18].

Soon after Chebyshev moved to [St. Petersburg](#), he was hired by Bunyakovski to work on the new edition of Euler’s works on the theory of numbers that had been undertaken by the Academy of Sciences. This edition (*L. Euleri Commentationes arithmeticae collectae*, 2 vols. [St. Petersburg, 1849]) comprised not only all of Euler’s previously published papers on the subject but also numerous manuscripts from the Academy’s archives; in addition Bunyakovski and Chebyshev contributed a valuable systematic review of Euler’s arithmetical works. Probably this work partly inspired Chebyshev’s own studies on the theory of numbers; these studies and the investigations of Chebyshev’s disciples advanced the theory of numbers in Russia to a level as high as that reached a century before by Euler. Some problems of the theory of numbers had been challenged by Chebyshev earlier, however, in his thesis *pro venia legendi*. He devoted to the theory of numbers his monograph *Teoria sravneny* (“Theory of Congruences”; 7), which he submitted for a doctorate in mathematics. He defended it at Petersburg University on 27 May 1849 and a few days later was awarded a prize for it by the Academy of Sciences. Chebyshev’s systematic analysis of the subject was quite independent and contained his own discoveries; it was long used as a textbook in Russian universities. It also contained the first of his two memoirs on the problem of distribution of prime numbers and other relevant problems; the second memoir, submitted to the Academy of Sciences in 1850, appeared in 1852. Through these two works, classics in their field, Chebyshev’s name became widely known in the scientific world. Later Chebyshev returned only seldom to the theory of numbers.

In 1850 Chebyshev was elected extraordinary professor of mathematics at Petersburg University; in 1860 he became a full professor. This was a decade of very intensive work by Chebyshev in various fields. First of all, during this period he began his remarkable studies on the theory of mechanisms, which resulted in the theory of the best approximation of functions. From his early years Chebyshev showed a bent for construction of mechanisms; and his studies at Moscow University stimulated his interest in technology, especially [mechanical engineering](#). In 1849–1851 he undertook a course of lectures on practical (applied) mechanics in the department of practical knowledge of Petersburg University (this quasiengineering department existed for only a few years); he gave a similar course in 1852–1856 at the Alexander Lyceum in Tsarskoe Selo (now Pushkin), near St. Petersburg. Chebyshev’s mission abroad, from July to November 1852, was another stimulus to his technological and mathematical work. In the evenings he talked with the best mathematicians of Paris, London, and Berlin or proceeded with his scientific work; morning hours were devoted to the survey of factories, workshops, and museums of technology. He paid special attention to steam engines and hinge-lever driving gears. He began to elaborate a general theory of mechanisms and in doing so met, according to his own words, certain problems of analysis that were scarcely known before (2, V, 249). These were problems of the theory of the best approximation of functions, which proved to be his outstanding contribution; in this theory his technological and mathematical inclinations were synthesized.

Back in St. Petersburg, Chebyshev soon submitted to the Academy of Sciences his first work on the problem of the best approximation of functions, prepared mainly during his journey and published in 1854. This was followed by another work on the subject, submitted in 1857 and published in 1859. These two papers marked the beginning of a great cycle of work in which Chebyshev was engaged for forty years. While in Europe, Chebyshev continued his studies on the integration of algebraic functions. His first published work on the problem, far surpassing the results at which he had arrived in the thesis *pro venia legendi*, appeared in 1853. Chebyshev published papers on this type of problem up to 1867, the object of them being to determine conditions for integration in the final form of different classes of irrational functions. Here, as in other cases, research was associated with university teaching; Chebyshev lectured on elliptic functions for ten years, until 1860.

In 1853 Chebyshev was voted an adjunct (i.e., junior academician) of the Petersburg Academy of Sciences with the chair of applied mathematics. Speaking for his nomination, Bunyakovski, Jacobi, Struve, and the permanent secretary of the Academy, P. N. Fuss, emphasized that Chebyshev’s merits were not restricted to mathematics; he had also done notable work in practical mechanics. In 1856 Chebyshev was elected an extraordinary academician and in 1859 ordinary academician (the highest academic rank), again with the chair of applied mathematics.

From 1856 Chebyshev was a member of the Artillery Committee, which was charged with the task of introducing artillery innovations into the Russian army. In close cooperation with the most eminent Russian specialists in ballistics, such as N. V. Maievski, Chebyshev elaborated mathematical devices for solving artillery problems. He suggested (1867) a formula for the range of spherical missiles with initial velocities within a certain limit; this formula was in close agreement with experiments. Some of Chebyshev's works on the theory of interpolation were the result of the calculation of a table of fire effect based on experimental data. Generally, he contributed significantly to ballistics.

Simultaneously Chebyshev began his work with the Scientific Committee of the Ministry of Education. Like Lobachevski, Ostrogradski, and a number of other Russian scientists, Chebyshev was active in working for the improvement of the teaching of mathematics, physics, and astronomy in secondary schools. For seventeen years, up to 1873, he participated in the elaboration of syllabi for secondary schools. His concise but solid reviews were of great value to the authors of textbooks that the Scientific Committee was supposed, as one of its principal functions, to approve or reject.

From the middle of the 1850's, the theory of the best approximation of functions and the construction of mechanisms became dominant in Chebyshev's work. Studies on the theory of functions embraced a very great diversity of relevant problems: the theory of orthogonal polynomials, the doctrine of limiting values of integrals, the theory of moments, interpolation, methods of approximating quadratures, etc. In these studies the apparatus of continued fractions, brilliantly employed by Chebyshev in many studies, was further improved.

From 1861 to 1888 Chebyshev devoted over a dozen articles to his technological inventions, mostly in the field of hinge-lever gears. Examples of these devices are preserved in the Mathematical Institute of the Soviet Academy of Sciences in Moscow and in the Conservatoire des Arts et Métiers in Paris.

In the 1860's Chebyshev returned to the theory of probability. One of the reasons for this new interest was, perhaps, his course of lectures on the subject started in 1860. He devoted only two articles to the theory of probability, but they are of great value and designate the beginning of a new period in the development of this field. In the article of 1866 Chebyshev suggested a very wide generalization of the law of large numbers. In 1887 he published (without extensive démonstration) a corresponding generalization of the central limit theorem of Moivre and Laplace.

Besides the above-mentioned mathematical and technological fields, which were of primary importance in Chebyshev's life and work, he showed lively interest in other problems of pure and applied mathematics. (His studies in cartography will be mentioned later). His paper of 1878, "Sur la coupe des vêtements" (Ia, II; 2, V), provided the basis for a new branch of the theory of surfaces. Chebyshev investigated a problem of binding a surface with cloth that is formed in the initial flat position with two systems of nonextensible rectilinear threads normal to one another. When the surface is bound with cloth the "Chebyshev net," whose two systems of lines form curvilinear quadrangles with equal opposite sides, appears. Wrapping a surface in cloth is a more general geometrical transformation than is deformation of a surface, which preserves the lengths of all the curved lines; distances between the points of the wrapped cloth that are situated on different threads are, generally speaking, changed in wrapping. In recent decades Chebyshev's theory of nets has become the object of numerous studies.

Theoretical mechanics also drew Chebyshev's attention. Thus, in 1884 he told Lyapunov of his studies on the problem of the ring-shaped form of equilibrium of a rotating liquid mass the particles of which are mutually attracted according to Newton's law. It is hard to know how far Chebyshev advanced in this field, for he published nothing on the subject. Still, the very problem of the form of equilibrium of a rotating liquid mass, which he proposed to Lyapunov, was profoundly investigated by the latter, who, along with Markov, was Chebyshev's most prominent disciple.

Among Chebyshev's technological inventions was a calculating machine built in the late 1870's. In 1882 he gave a brief description of his machine in the article "Une machine arithmétique à mouvement continu" (Ia, II; 2, IV). The first model (*ca.* 1876) was intended for addition and subtraction; he supplemented it with an apparatus enabling one to multiply and divide as well. Examples of the machine are preserved in the Muséum of History in Moscow and in the Conservatoire des Arts et Métiers in Paris.

During this period Chebyshev was active in the work of various scientific societies and congresses. Between 1868 and 1880 he read twelve reports at the congresses of Russian naturalists and physicians, and sixteen at the sessions of the Association Française pour l'Avancement des Sciences between 1873 and 1882; it was at these sessions that he read "Sur la coupe des vêtements" and reported on the calculating machine. He gave numerous démonstrations of his technological inventions both at home and abroad. Chebyshev was in contact with the Moscow and St. Petersburg mathematical societies and with the Moscow Technological College (now Bauman Higher Technological College).

In the summer of 1882, after thirty-five years of teaching at Petersburg University, Chebyshev retired from his professorship, although he continued his work at the Academy of Sciences. Nonetheless, he was constantly in touch with his disciples and young scientists. He held open house once a week, and K. A. Posse states that "hardly anybody left these meetings without new ideas and encouragement for further endeavor" (2, V, 210); it was sufficient only that the problem be relevant to the fields in which Chebyshev had been interested. When Chebyshev was over sixty, he could not work at his former pace; nevertheless, he published some fifteen scientific papers, including a fundamental article on the central limit theorem. He submitted his last work to the Academy of Sciences only a few months before his death at the age of seventy-three.

Besides being a member of the Petersburg Academy of Sciences, Chebyshev was elected a corresponding (1860) and—the first Russian scientist to be so honored—a foreign member (1874) of the Academy of Sciences of the Institut de France, a corresponding member of the Berlin Academy of Sciences (1871), a member of the Bologna Academy (1873), and a foreign member of the [Royal Society](#) of London (1877), of the Italian Royal Academy (1880), and of the Swedish Academy of Sciences (1893). He was also an honorary member of all Russian universities and of the Petersburg Artillery Academy. He was awarded numerous Russian orders and the French Legion of Honor.

Petersburg Mathematical School. Chebyshev's importance in the history of science consists not only in his discoveries but also in his founding of a great scientific school. It is sometimes called the Chebyshev school, but more frequently the Petersburg school because its best-known representatives were almost all educated at Petersburg University and worked either there or at the Academy of Sciences. The Petersburg mathematical school owes its existence partly to the activity of Chebyshev's elder contemporaries, such as Bunyakovski and Ostrogradski; nevertheless, it was Chebyshev who founded the school, directed and inspired it for many years, and influenced the trend of mathematics teaching at Petersburg University. For over half a century mathematical chairs there were occupied by Chebyshev's best disciples or their own disciples. Thus the mathematics department of Petersburg University achieved a very high academic level. Some disciples of Chebyshev took his ideas to other Russian universities.

Chebyshev was highly endowed with the ability to attract beginners to creative work, setting them tasks demanding profound theoretical investigation to solve and promising brilliant results. The Petersburg school included A. N. Korokin, Y. V. Sohotski, E. I. Zolotarev, A.A. Markov, A.M. Lyapunov, K.A. Posse, D.A. Grave, G.F. Voronoi, A. V. Vassiliev, V.A. Steklov, and A.N. Krylov. Chebyshev, however, was such a singular individual that he also influenced scientists who did not belong to his school, both in Russia (e.g., N.Y. Sonin) and abroad. During the latter half of the nineteenth and the beginning of the twentieth centuries the Petersburg mathematical school was one of the most prominent schools in the world and the dominant one in Russia. Its ideas and methods formed an essential component of many divisions of pure and applied mathematics and still influence their progress.

Although a great variety of scientific trends were represented in the Petersburg mathematical school, the work of Chebyshev and his followers bore some important common characteristics. Half seriously, Chebyshev shortly before his death said to A.V. Vassiliev that previously mathematics knew two periods: during the first the problems were set by gods (the Delos problem of the duplication of a cube) and during the second by demigods, such as Fermat and Pascal. "We now entered the third period, when the problems were set by necessity" (16, p. 59). Chebyshev thought that the more difficult a problem set by scientific or technological practice, the more fruitful the methods suggested to solve it would be and the more profound a theory arising in the process of solution might be expected to be.

The unity of theory and practice was, in Chebyshev's view, the moving force of mathematical progress. He said in a speech entitled "Cherchenie geograficheskikh kart" ("Drawing Geographical Maps"), delivered at a ceremonial meeting of Petersburg University in 1856:

Mathematical sciences have attracted especial attention since the greatest antiquity; they are attracting still more interest at present because of their influence on industry and arts. The agreement of theory and practice brings most beneficial results; and it is not exclusively the practical side that gains; the sciences are advancing under its influence as it discovers new objects of study for them, new aspects to exploit in subjects long familiar. In spite of the great advance of the mathematical sciences due to works of outstanding geometers of the last three centuries, practice clearly reveals their imperfection in many respects; it suggests problems essentially new for science and thus challenges one to seek quite new methods. And if theory gains much when new applications or new developments of old methods occur, the gain is still greater when new methods are discovered; and here science finds a reliable guide in practice [2, V, 150].

Among scientific methods important for practical activity Chebyshev especially valued those necessary to solve the same general problem:

How shall we employ the means we possess to achieve the maximum possible advantage? Solutions of problems of this kind form the subject of the so-called theory of the greatest and least values. These problems, which are purely practical, also prove especially important for theory: all the laws governing the movement of ponderable and imponderable matter are solutions of this kind of problem. It is impossible to ignore their special influence upon the advance of mathematical sciences [*ibid.*].

These statements (reminding one somewhat of Euler's ideas on the universal meaning of the principle of least action) are illustrated by Chebyshev with examples from the history of mathematics and from his own works on the theory of mechanisms and the theory of functions. The speech quoted above is specially devoted to solution of the problem of searching for such conformal projections upon a plane of a given portion of the earth's surface under which change in the scale of image (different in various parts of a map) is the least, so that the image as a whole is the most advantageous. The projection sought bears a characteristic particularity: on the border of the image the scale preserves the same value. Démonstration of this theorem of Chebyshev's was first published by Grave in 1894.

Chebyshev's general approach to mathematics quite naturally resulted in his aspiration toward the effective solution of problems and the discovery of algorithms giving either an exact numerical answer or, if this proved impossible, an approximation ready for scientific and practical applications. He interpreted the strictness of the theory in the event of

approximate evaluations as a possibility of precise definition of limits not trespassed by the error of approximation. Chebyshev was a notable representative of the “mathematics of inequalities” of the latter half of the nineteenth century. His successors held similar views.

Lastly, not the least characteristic feature of Chebyshev’s works from the early period on was his inclination toward possibly elementary mathematical apparatus, particularly his almost exclusive use of functions in the real domain. He was especially adept at using continued fractions. Many other scholars of his school made wider use of contemporary analysis, especially of the theory of functions of complex variables.

The Petersburg school was concerned with quite a number of subjects. It dealt primarily with the domains of pure and applied mathematics investigated by Chebyshev himself and developed by his disciples. However, Chebyshev’s followers worked in other areas of mathematical sciences that were far from the center of Chebyshev’s own interests. Growing in numbers and ability, the Petersburg school gradually became an aggregate of scientific schools brought together by similar principles of study and closely connected both on the level of ideas and on the personal level; they differed mainly in their predominant mathematical subjects: the theory of numbers, the theory of the best approximation of functions, the theory of probability, the theory of differential equations, and mathematical physics.

Although the work of Chebyshev and his school was independent in the formulation of numerous problems and in the elaboration of methods to solve them and discovered large new domains of mathematical study, it was closely related to mathematics of the eighteenth and the first half of the nineteenth centuries. In Russia the school developed the tradition leading back to Euler, whose works were thoroughly studied and highly valued. Much as they differed as individuals (the difference in mathematics of their respective epoches was no less great), Chebyshev and Euler had much in common. Both were interested in a great variety of problems, from the theory of numbers to mechanical engineering. Both were aware of the profound connection of mathematical theory with its applications and tended to set themselves concrete problems as a source of theoretical conclusions that were later generalized; both, on the other hand, understood the vital necessity of developing mathematics in its entirety, including problems the solution of which did not promise any immediate practical gain. Finally, both were always seeking most effective solutions that approached computing algorithms.

It is important to note that at the beginning of the present century younger representatives of the Chebyshev schools started to bring about its contact with other trends in mathematics, which soon led to great progress.

Theory of Numbers. It had been proved in ancient Greece that there exist infinitely many prime numbers 2,3,5,7,... This principal result in the doctrine of the distribution of prime numbers remained an isolated result until the end of the eighteenth century, when the first step in investigation of the frequency of prime numbers in the natural number series 1,2,3,4,... was made: Legendre suggested in 1798–1808 the approximate formula

to express the number of prime numbers not exceeding a given number x , e.g., for the function designated $\pi(x)$. This formula accorded well with the table of prime numbers from 10,000 to 1,000,000. In his article “Sur la fonction qui détermine la totalité des nombres premiers inférieurs à une limite donnée” (1849; see Ia, I; 2,1) Chebyshev, making use of the properties of Euler’s zeta function in the real domain, proved the principal inaccuracy of Legendre’s approximate formula and made a considerable advance in the study of the properties of the function $\pi(x)$. According to Legendre’s formula, the difference $x/\pi(x)$; — $\ln x$ for $x \rightarrow \infty$ has the limit — 1.08366. Chebyshev demonstrated that this difference cannot have a limit differing from —1. With sufficiently great x the integral gives better approximations to $\pi(x)$ than Legendre’s and similar formulas; besides, the difference

inconsiderable within the limits of the tables used by Legendre, reaches the minimum for $x \approx 1,247,689$ and then increases without limit with the increase of x .

It also followed from Chebyshev’s theorems that the ratio of the function $\pi(x)$ to $\int_2^x dx/\ln x$ cannot, for $x \rightarrow \infty$ have a limit differing from unity.

Chebyshev later continued the study of the properties of $\pi(x)$ In his “Mémoire sur les nombres premiers” (1850, pub. 1852; see Ia, I; 2, II) he demonstrated that $\pi(x)$ can differ from $x/\ln x$ by no more than approximately 10 percent—more exactly, that

$$0.92129x/\ln x < \pi(x) < 1.10555x/\ln x$$

(although he stated this result in a slightly different form)

Other remarkable discoveries were described in these two articles. The second article demonstrates Bertrand’s conjecture (1845) that for $n > 3$, between n and $2n - 2$ there is always at least one [prime number](#). In it Chebyshev also proved some theorems on convergence and on the approximate calculation of the sums of infinite series the members of which are functions of successive prime numbers (the first series of this kind were studied by Euler). In a letter to P. N. Fuss published in 1853 (Ia, 1; 2, II) Chebyshev set a problem of estimating the number of prime numbers in arithmetical progressions and gave some results concerning progressions with general members of the form $4n+1$ and $4n+3$.

Chebyshev's studies on the distribution of prime numbers were developed by numerous scientists in Russia and abroad. Important advances were made in 1896 by Hadamard and Vallée-Poussin, who, employing analytical functions of a complex variable, proved independently the asymptotic law of distribution of prime numbers:

Studies in this field of the theory of numbers are being intensively conducted.

Among Chebyshev's other works on the theory of numbers worthy of special attention is his article "Ob odnom arifmeticheskom voprose" ("On One Arithmetical Problem," 1866; Ia, I:2, II), which served as a point of departure for a series of studies devoted to a linear heterogeneous problem of the theory of diophantine approximations. The problem, in which Hermite, Minkowski, Remak, and others were later interested, was completely solved in 1935 by A. Y. Khintchine.

Integration of Algebraic Functions. Chebyshev's studies on the integration of algebraic functions were closely connected with the work of Abel, Liouville, and, in part, Ostrogradski. In the article "Sur l'integration des différentielles irrationnelles" (1853; see Ia, I:2, II) Chebyshev succeeded in giving a complete solution of the problem of defining the logarithmic part of the integral

for the case when it is expressed in final form—here the functions $f(x)$, $F(x)$, $\theta(x)$ are integral and rational, and m is any positive integer. But the article is known mostly for the final solution of the problem of the integration of the binomial differential $x^m(a + bx^n)^p dx$ it contains; here m , n , p are rational numbers. Generalizing Newton's result, Goldbach and Euler showed that this type of integral is expressible in elementary functions in any of the three cases when p is an integer; $(m + 1)/n$ is an integer; and $(m + 1)/n + p$ is an integer. Chebyshev demonstrated that the three cases are the only cases when the integral mentioned mentioned is taken in the final form. This theorem is included in all textbooks on [integral calculus](#).

In the theory of elliptic integrals Chebyshev substantially supplemented Abel's results. Integration of any elliptic differential in the final form is reduced to integration of a fraction that has a linear function $x + A$ in the numerator and a square root of a polynomial of the fourth degree in the denominator. Chebyshev considered the problem in the article "Sur l'integration de la différentielle

(1861; see Ia, I; 2, II). It is supposed that the polynomial in the denominator has no multiple roots (in the case of multiple roots the differential is integrated quite easily). The elliptic differential in question either is not integrable in elementary functions or is integrable for one definite value of the constant A . Abel had shown that the latter case occurred if a continued fraction formed by expansion of

is periodic. However, Abel could offer no final criterion enabling one to ascertain nonperiodicity of such an expansion. Chebyshev gave a complete and efficient solution of the problem with rational numbers $\alpha, \beta, \gamma, \delta$: he found a method enabling one to ascertain nonperiodicity of such an expansion, by means of the finite number of operations, which in turn depends on the finite number of integer solutions of a system of two equations with three unknowns; he also determined the limit of the number of operations necessary in case of integrability. He did not publish any complete démonstration of his algorithm; this was done by Zolotarev in 1872. Soon Zolotarev suggested a solution of the same problem for any real coefficients $\alpha, \beta, \gamma, \delta$, for which purpose he devised his own variant of the theory of ideals.

Chebyshev also studied the problem of integrability in the final form of some differentials containing a cube root of a polynomial. In Russia this direction of study was followed by I.L. Ptashitski and I. P. Dolbnia, among others.

Theory of the Best Approximation of Functions. It was said that Chebyshev had approached the theory of best approximation of functions from the problems of the theory of hinge mechanisms, particularly from the study of the so-called Watt parallelogram employed in steam engines and other machines for the transformation of rotating movement into rectilinear movement. In fact, it is impossible to obtain strictly rectilinear movement by this means, which produces a destructive effect. Attempting somehow to reduce the deviation of the resultant movement from the rectilinear, engineers searched empirically for suitable correlation between the parts of mechanisms. Chebyshev set the task of elaborating a sound theory of the problem, which was lacking; he also devised a number of curious mechanisms that, although they could not strictly secure rectilinear movement, could compete with "strict" mechanisms because the deviation was very small. Thus, the virtually exact seven-part rectifying device suggested by Charles Peaucellier and independently by L. I. Lipkin is, in view of the complexity of construction, less useful in practice than Chebyshev's four-part device described in the article "Ob odnom mekhanizme" ("On One Mechanism"; 1868; see Ia, II; 2, IV).

Chebyshev made a profound investigation of the elements of a hinge mechanism, setting out to achieve the smallest deviation possible of the trajectory of any points from the straight line for the whole interval studied. A corresponding mathematical problem demanded that one choose, from among the functions of the given class taken for approximation of the given function, that function with which the greatest modulo error is the smallest under all considered values of the argument. Some special problems of this kind had previously been solved by Laplace, Fourier, and Poncelet. Chebyshev laid foundations for a general theory proceeding from the approximation of functions by means of polynomials. The problem of approximation of the given function by means of polynomials might be formulated differently. Thus, in an expansion of the given function $f(x)$ into a Taylor series of powers of the difference $x - a$, the sum of the first $n + 1$ members of the series is a polynomial of n th degree that in the neighborhood of the value $x = a$ gives the best approximation among all polynomials of the same degree. Chebyshev

set the task of achieving not a local best approximation but the uniform best approximation throughout the interval; his object was to find among all polynomials $p_n(x)$ of n th degree such a polynomial that the maximum $|f(x) - P_n(x)|$ for this interval is the smallest.

In the memoir “Théorie des mécanismes connus sous le nom de parallélogrammes” (1854; see Ia, I; 2, II), which was the first in a series of works in this area, Chebyshev considered the problem of the best approximation of the function $f(x) = x^n$ by means of polynomials of degree $n - 1$; that is, he considered the problem of the determination of the polynomial of the n th degree, $x^n + p_1x^{n-1} + \dots + P_n$ with the leading coefficient equal to unity, deviating least from zero. This formulation of the problem engendered the frequently used term “theory of polynomials deviating least from zero” (the term was used by Chebyshev himself). In the case of the interval $(-1, +1)$ this polynomial is

and its maximum deviation from zero is $1/2^{n-1}$. Polynomials $T_n(x)$, named for Chebyshev, form an orthogonal system with respect to a weight function

In his next long memoir, “Sur les questions de minima qui se rattachent à la représentation approximative des fonctions” (1859; see Ia, I; 2, II), Chebyshev extended the problem to all kind of functions $F(x, P_1, P_2, \dots, P_n)$ depending on n parameters, expressed general views concerning the method of solution, and gave a complete analysis of two cases when F is a rational function; in this work some curious theorems on the limits of real roots of algebraic equations were obtained. Varying restrictions might be imposed upon the function of the best approximation that is to be determined; Chebyshev also solved several problems of this type.

Gradually considering various problems either directly relevant to the theory of the best approximation of functions or connected with it, Chebyshev obtained important results in numerous areas.

(a) The theory of interpolation and, especially, interpolation on the method of least squares (1855–1875).

(b) The theory of orthogonal polynomials. Besides polynomials $T_n(x)$ deviating least from zero, Chebyshev in 1859 proceeded from consideration of different problems to the study of other orthogonal systems, such as Hermite and Laguerre polynomials. However, he did not take up the determination of polynomials under the condition of orthogonality in the given interval with respect to the given weight function; he introduced polynomials by means of an expansion in continued fractions of certain integrals of the

type, where $p(x)$ is the weight function.

(c) The theory of moments was first treated by Chebyshev in his article “Sur les valeurs limites des intégrales” (1874; see Ia, 1; 2, III). Here the following problem is considered: given the values of moments of different orders of an unknown function $f(x)$ e.g., of the integrals

in an interval (A, B) where $f(x) > 0$, one is required to find the limits within which the value of the integral

lies ($A < a < b < B$). Chebyshev once more connected the investigation of the problem with expansion of the integral

in continued fraction; in conclusion he gave a detailed solution of the problem for a special case $m = 2$ formulated in mechanical interpretation: “Given length, weight, site of the center of gravity, and the [moment of inertia](#) of a material straight line with an unknown density that changes from one point to another, one is required to find the closest limits with respect to the weight of a certain segment of this straight line” (2, III, 65). Chebyshev’s work on estimations of integrals received important application in his studies on the theory of probability.

(d) Approximate calculus of definite integrals. In his article “Sur les quadratures” (1874; Ia, II; 2, III) Chebyshev, proceeding from work by Hermite, suggested new general formulas of approximate quadratures in which all the values of the integrand are introduced under the same coefficient or, at least, under coefficients differing only in sign. The aforementioned characteristic of the coefficients renders Chebyshev’s quadrature formulas very suitable for the calculus in many cases; A.N. Krylov used them in his studies on the theory of ships. One of the formulas is

the values x_1, x_2, x_3, \dots are obtained from an equation of the n th degree: . Chebyshev himself calculated these values for $n = 2, 3, 4, 5, 6, 7$; for $n = 8$ the equation has no real roots and the formula is unusable; for $n = 9$ the roots are real again but, as S. N. Bernstein demonstrated in 1937, beginning at $n = 10$ the formula is again unusable.

This large cycle of Chebyshev’s works was carried further by Zolotarev, Markov, Sonin, Posse, Steklov, Stieltjes, Riesz, and many others; work in these and new fields is being continued. It is worth special mention that at the beginning of the twentieth century the theory of the best approximation acquired essentially new features through the connections between Chebyshev’s ideas and methods, on the one hand, and those developed in western Europe, on the other. Works of S. N. Bernstein and his disciples were of primary importance here.

Theory of Probability. In his article “O srednikh velichinakh” (On Mean Values”; 1866; see Ia, 1; 2, II) Chebyshev, using an inequality previously deduced by J. Bienayme’, gave a precise and very simple démonstration of the generalized law of large numbers that might be thus expressed in modern terms: If x_1, x_2, x_3, \dots are mutually independent in pairs of random quantities, with expectation values a_1, a_2, a_3, \dots and dispersions b_1, b_2, b_3, \dots , the latter being uniformly limited—e.g., all $b_n \leq C$ —then for any $\epsilon > 0$ the probability P of an inequality

is $\geq 1 - (C/n\epsilon^2)$. From this it follows immediately that

The theorems of Poisson and Jakob Bernoulli are only special cases of Chebyshev’s law of large numbers for sequences of random quantities.

Developing his method of moments and of estimation of the limit values of integrals, Chebyshev also managed to extend to sequences of independent random quantities the central limit theorem of Moivre and Laplace: within the framework of former suppositions, supplemented with the condition that there exist expectation values (moments) of any order and that they all are uniformly limited,

This theorem and the draft of its démonstration were published by Chebyshev in the article “O dvukh teoremakh otноситelno veroyatnostey” (“On Two Theorems Concerning Probability”; 1887; see Ia, II; 2, III); the first theorem mentioned in the title is the law of large numbers. Chebyshev’s second theorem enabled one to apply, on a larger scale, the theory of probability to mathematical statistics and natural sciences; both regard the phenomenon under study as resulting from common action of a great number of random factors, each factor displaying considerably smaller influence independently in comparison with their influence as a set. According to this theorem, such common action closely follows the normal distribution law. Chebyshev’s démonstration was supplemented a decade later by Markov.

Chebyshev’s studies on limit theorems were successfully developed by his disciples and successors, first by Markov, Lyapunov, and Bernstein and later by numerous scientists. A. N. Kolmogorov, a foremost authority in the field, says:

From the standpoint of methodology, the principal meaning of the radical change brought about by Chebyshev is not exclusively that he was the first mathematician to insist on absolute accuracy in démonstration of limit theorems (the proofs of Moivre, Laplace, and Poisson were not wholly consistent on the formal logical grounds in which they differ from those of Bernoulli, who demonstrated his limit theorem with exhaustive arithmetical accuracy). The principal meaning of Chebyshev’s work is that through it he always aspired to estimate exactly in the form of inequalities absolutely valid under any number of tests the possible deviations from limit regularities. Further, Chebyshev was the first to estimate clearly and make use of the value of such notions as “random quantity” and its “expectation (mean) value.” These notions were known before him; they are derived from fundamental notions of the “event” and “probability.” But random quantities and their expectation values are subject to a much more suitable and flexible algorithm [21, p. 56].

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Chebyshev’s works are the following:

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(Ia) *Oeuvres de P. L. Tchebychef*, A.A. Markov and N. Y. Sonin, eds., 2 vols. (St. Petersburg, 1899–1907), a French version of (1). Both (Ia) and (1) contain a biographical note based entirely on (9).

(2) *Polnoe sobranie sochineny*, 5 vols. (Moscow-Lenin-grad, 1944–1951): I, *Teoria chisel*; II–III, *Matematicheskyy analiz*; IV, *Teoria mekhanizmov*; V, *Prochie sochinenia. Biograficheskie materialy*. This ed. contains very valuable scientific commentaries that are largely completed and developed in (3).

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