ENDOMORPHISMS OF THE SEMIGROUP OF ORDER-PRESERVING MAPPINGS

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ABSTRACT. We characterize the endomorphisms of the semigroup of all order-preserving mappings on a finite chain. We show that there are three types of endomorphism: automorphisms, constants, and a certain type of endomorphism with two idempotents in the image.

1. INTRODUCTION & THE MAIN THEOREM

A mapping $f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ is called *order-preserving* if $(i)f \leqslant (j)f$ whenever $i \leqslant j$. Throughout, we will write functions to the right of their argument and compose them from left to right. The semigroup of all order-preserving mappings from $\{1, 2, \ldots, n\}$ to itself under composition of functions is denoted O_n. The semigroup O_n has been extensively studied by many authors since the 1960s. The identity of O_n is denoted by 1_n. An *endomorphism* ϕ of a semigroup S is a mapping $\phi : S \longrightarrow S$ such that $(x)\phi(y)\phi = (xy)\phi$ for all $x, y \in S$. We will denote the semigroup of endomorphisms of S by $\mathrm{End}(S)$. A bijective endomorphism is called an *automorphism*. In this note we completely describe the endomorphisms of O_n for all $n \in \mathbb{N}$, specify the number of these endomorphisms, and describe the structure of the semigroup $\mathrm{End}(O_n)$.

In 1962, Aĭzenštat [1] gave a presentation for O_n from which it can be deduced that the only non-trivial automorphism of O_n where $n > 1$ is that given by conjugation by the permutation $\sigma = (1\,n)(2\,n-1)\cdots(\lfloor n/2 \rfloor\, \lceil n/2 \rceil + 1)$. In other words, if ϕ is a non-identity automorphism of O_n and $n > 1$, then $(s)\phi = \sigma^{-1}s\sigma$ for all $s \in O_n$. We will write f^σ to denote $\sigma^{-1}f\sigma$. Denote the set of idempotents of O_n by $E(O_n)$.

Finding the automorphisms and endomorphisms of transformation semigroups is a classical problem; they have been determined for many classical transformation semigroups. For example, the full transformation semigroup [11], the symmetric inverse semigroup [10], and the Brauer-type semigroups [9].

The endomorphisms of O_n are described in the following theorem.

Theorem 1.1. *Let $\phi : O_n \to O_n$ be any mapping. Then ϕ is an endomorphism of O_n if and only if one of the following holds:*

(a) *ϕ is an automorphism and so ϕ is the identity or $\phi : s \mapsto s^\sigma$ where $\sigma = (1\,n)(2\,n-1)\cdots(\lfloor n/2 \rfloor\, \lceil n/2 \rceil + 1)$ for all $s \in O_n$;*

(b) *there exist $e, f \in E(O_n)$ with $e \neq f$ and $ef = fe = f$ such that $1_n\phi = e$ and $(O_n \setminus \{1_n\})\phi = f$;*

(c) *ϕ is a constant mapping with idempotent value.*

We will denote the non-identity automorphism of O_n by α and the endomorphisms satisfying (b) or (c) above by $\phi_{e,f}$ or ϕ_e, respectively, where $1_n\phi_{e,f} = e$ and $(O_n \setminus \{1_n\})\phi_{e,f} = \{f\}$ and $O_n\phi_e = \{e\}$.

Theorem 1.2. *If $n > 1$, then $|\operatorname{End}(O_n)| = 2 + \sum_{i=0}^{n-1} \binom{n+i}{2i+1} F_{2i+2}$ where F_{2i+2} denotes the $(2i+2)$th Fibonacci number.*

Theorem 1.3. *The $\mathcal{L}$-classes and $\mathcal{H}$-classes of $\operatorname{End}(O_n)$ coincide and the only non-trivial such class is $\operatorname{Aut}(O_n)$.*

The $\mathcal{R}$-classes and $\mathcal{D}$-classes of $\operatorname{End}(O_n)$ coincide and are $\operatorname{Aut}(O_n)$, $\{\phi_{1_n,e} : e \neq 1_n\}$, $\{\phi_{e,f}, \phi_{e,f}\alpha\}$ where $ef = fe = f$, $f \neq e$ and $e \neq 1_n$, and $\{\ \phi_e\ :\ e \in E(O_n)\ \}$.

The $\mathcal{R}$-classes of the form $\{\phi_{e,f}, \phi_{e,f}\alpha\}$ from Theorem 1.3 either have one or two elements depending on whether $\phi_{e,f} = \phi_{e,f}\alpha$ or not. The number of $\mathcal{R}$-classes with one element is given in the next theorem.

Theorem 1.4. *The number of singleton $\mathcal{R}$-classes in $\operatorname{End}(O_n)$ equals*

(i) $\sum_{i=0}^{k-1} (F_{2i+2} - 1)\binom{k+i}{2i+1} - (F_{2k} - 1)$ *if $n = 2k$;*

(ii) $\sum_{i=0}^{k} \sum_{j=1}^{k-i+1} (F_{2j} - 1)\binom{k-i+j-2}{2j-3} - (F_{2(k+1)} - 1)$ *if $n = 2k+1$;*

where F_r denotes the rth Fibonacci number and $\binom{r}{-1} = 1$.

Corollary 1.5. *The number of non-singleton $\mathcal{R}$-classes in $\operatorname{End}(O_n)$ equals*

(i) $\sum_{i=0}^{n-1} F_{2i+2}\binom{n+i}{2i+1} - \sum_{i=0}^{k-1} (F_{2i+2} - 1)\binom{k+i}{2i+1} - (2F_{2n} + F_{2k} - 2)$ *if $n = 2k$;*

(ii) $\sum_{i=0}^{n-1} F_{2i+2}\binom{n+i}{2i+1} - \sum_{i=0}^{k} \sum_{j=1}^{k-i+1} (F_{2j} - 1)\binom{k-i+j-2}{2j-3} - (2F_{2n} + F_{2(k+1)} - 2)$ *if $n = 2k+1$;*

where F_r denotes the rth Fibonacci number.

We note that if $n \geqslant 3$, then there exist idempotents $e, f \in O_n$ such that $ef = fe = f$, $f \neq e$ and $e \neq 1_n$. Hence the $\mathcal{L}$-class $\{\phi_{e,f}\}$ (in $\operatorname{End}(O_n)$) does not contain idempotents. Therefore the semigroup $\operatorname{End}(O_n)$ is not regular when $n \geqslant 3$.

The remainder of the note is dedicated to proving the above theorems. To do so we require the following notions. The *image* of an element $f \in O_n$ is denoted by $\operatorname{im}(f)$ and the *kernel* of f is the equivalence relation $\{\ (x, y) \in \{1, \ldots, n\} \times \{1, \ldots, n\} : xf = yf\ \}$ denoted by $\ker(f)$. The *rank* of an element $f \in O_n$ is $|\operatorname{im}(f)|$ and denoted $\operatorname{rank}(f)$. Some of the important properties of O_n that we require later are: it is regular (for all $f \in O_n$ there exists $g \in O_n$ such that $fgf = f$), its Green's relations are described by

$$\begin{array}{lll} f\mathcal{L}g & \text{if and only if} & \operatorname{im}(f) = \operatorname{im}(g) \\ f\mathcal{R}g & \text{if and only if} & \ker(f) = \ker(g) \\ f\mathcal{D}g & \text{if and only if} & \operatorname{rank}(f) = \operatorname{rank}(g) \\ f\mathcal{H}g & \text{if and only if} & f = g, \end{array}$$

and the number of idempotent elements is the $2n$ th Fibonacci number F_{2n}. Further information regarding O_n can be found in [4] and [5] and regarding semigroups, in general, can be found in [6]. We denote the $\mathcal{D}$-class of those elements in O_n with rank k by D_k.

It is well-known that I is an ideal of O_n if and only if $I = \{f \in O_n : \text{rank}(f) \leqslant k\}$ for some $1 \leqslant k \leqslant n$; for a proof see [3]. In 1962, Aĭzenštat [2] showed that the non-trivial congruences of O_n are exactly those where the only non-singleton class is $I_k = \{\, f \in O_n \ :\ \text{rank}(f) \leqslant k \,\}$ for some $1 \leqslant k \leqslant n$. Another proof of this can be found in [8].

2. PROOFS OF THE THEOREMS

It is straightforward to verify that the mappings described in Theorem 1.1 are endomorphisms of O_n. So, it remains to prove that there are no further endomorphisms. Throughout the remainder of the note we will assume that $n > 1$.

Let ϕ be an endomorphism of O_n. If ϕ is an automorphism of O_n or a constant mapping, then ϕ satisfies (a) or (c) in Theorem 1.1. So, we may assume that ϕ is not an automorphism and not constant. From the comments in the introduction, there exists $1 \leqslant k \leqslant n-1$ such that the unique non-singleton kernel class of ϕ is I_k. If $k = n-1$, then ϕ is of type (b) from Theorem 1.1. Hence we may assume that $1 \leqslant k \leqslant n-2$. In the following lemmas we will prove that this is not possible and so conclude the proof.

Note that ϕ has only singleton kernel classes on $O_n \backslash I_k$ and so ϕ must be injective on $O_n \backslash I_k$. The unique element of $I_k\phi$ is an idempotent. Throughout the remainder of the note we will denote this idempotent by f.

Lemma 2.1. *Let $g, h \in D_i$ where $k+1 \leqslant i \leqslant n-1$. Then $g\mathcal{R}h$ if and only if $g\phi\mathcal{R}h\phi$. Likewise, $g\mathcal{L}h$ if and only if $g\phi\mathcal{L}h\phi$.*

Proof. We prove the theorem only for Green's $\mathcal{R}$-relation, the proof for Green's $\mathcal{L}$-relation is analogous.

($\Rightarrow$) Since ϕ is a homomorphism, this implication is immediate.

($\Leftarrow$) Let $g\phi\mathcal{R}h\phi$. As I_i is a regular subsemigroup of O_n, it follows that $I_i\phi$ is a regular subsemigroup of O_n. Thus, from [6, Proposition 2.4.2], $\mathcal{R}^{I_i\phi} = \mathcal{R} \cap (I_i\phi \times I_i\phi)$. Thus $g\phi\mathcal{R}^{I_i\phi}h\phi$ and so there exist $a, b \in I_i$ with $h\phi = g\phi \cdot a\phi$ and $g\phi = h\phi \cdot b\phi$.

If $\text{rank}(ga) \leqslant k$, then $(ga)\phi = f$ and so $h\phi = g\phi \cdot a\phi = (ga)\phi = f$, and so $h \in (f)\phi^{-1} = I_k$, a contradiction. Hence $\text{rank}(ga) > k$ and likewise $\text{rank}(hb) > k$. It follows, since ϕ is injective on $O_n \setminus I_k$, that $h = ga$ and $g = hb$. In other words, $g\mathcal{R}h$. □

Lemma 2.2. *$D_{k+1}\phi \subseteq D_l$ where* $\text{rank}(f) < l < k+1$.

Proof. Homomorphisms of semigroups preserve $\mathcal{D}$-classes, and so $D_{k+1}\phi \subseteq D_l$ for some l. Since ϕ is injective on $D_{k+1} \cup D_{k+2} \cup \cdots \cup D_n$ and ϕ preserves the partial order on $\mathcal{D}$-classes, it follows that $l \leqslant k+1$.

Seeking a contradiction assume that $\text{rank}(f) > k$. Then $(I_k)\phi \subseteq D_{k+1} \cup D_{k+2} \cup \cdots \cup D_n = O_n \setminus I_k$. Again since ϕ preserves the partial order of $\mathcal{D}$-classes, $(O_n \setminus I_k)\phi \subseteq O_n \setminus I_k$. Since ϕ is injective on $O_n \setminus I_k$, it follows that $(O_n \setminus I_k)\phi = O_n \setminus I_k$. Hence there exists $g \in O_n \setminus I_k$ such that $g\phi = h\phi$ for any $h \in I_k$, a contradiction, since the kernel class of g is singleton. Therefore $\text{rank}(f) \leqslant k$ and so $f\phi = f$ and $\text{rank}(f) \leqslant l$.

Assume that $l = k + 1$. Then $D_{k+1}\phi = D_{k+1}$, since $\phi|_{D_{k+1}}$ is injective. Let $d \in D_{k+1}$ with $fd \neq f$. Note that $d\phi^{-1}$ is a set containing a single element and that element is contained in D_{k+1}. Hence, remembering that $\operatorname{rank}(f) \leqslant k$,

$$f \neq f \cdot d = f\phi \cdot d = (f \cdot d\phi^{-1})\phi = f,$$

a contradiction. Therefore $l < k + 1$, as required.

Finally, assume that $\operatorname{rank}(f) = l$ and let $g \in D_{k+1}$. Then $g\phi\mathcal{D}f$. Since

$$g\phi \cdot f = g\phi \cdot f\phi = (gf)\phi = f = (fg)\phi = f\phi \cdot g\phi = f \cdot g\phi$$

it follows that $\operatorname{im}(f) = \operatorname{im}(g\phi)$ and $\ker(f) = \ker(g\phi)$. Thus $f\mathcal{H}g\phi$ and so $f\phi = f = g\phi$, contradicting the assumption that f and g are in different kernel classes of ϕ. □

Lemma 2.3. $k \geqslant \lfloor n/2 \rfloor$.

Proof. By Lemma 2.1, the number of $\mathcal{L}$-classes in $D_{k+1}\phi$ is $\binom{n}{k+1}$. But, by Lemma 2.2, $D_{k+1}\phi \subseteq D_l$ where $l < k + 1$ and the number of $\mathcal{L}$-classes in D_l is $\binom{n}{l}$. Hence $\binom{n}{k+1} \leqslant \binom{n}{l}$ and so $k \geqslant \lfloor n/2 \rfloor$. □

The following lemma is straightforward but we include a proof for the sake of completeness.

Lemma 2.4. *There exist idempotents* $e_1, \ldots, e_{n-1} \in D_{k+1}$ *such that either* $e_ie_j \in I_k$ *or* $e_je_i \in I_k$ *for all* $1 \leqslant i < j \leqslant n - 1$.

Proof. Let $g_i, h_i \in O_n$ be defined by

$$(j)g_i = \begin{cases} i & j \in \{i, i+1, \ldots, i+n-k-1\} \\ j & j < i \text{ or } j > i+n-k-1 \end{cases}$$

for all $1 \leqslant i \leqslant k + 1$ and

$$(j)h_i = \begin{cases} i & j \in \{i, i+1\} \\ i+2 & j \in \{i+2, i+3, \ldots, i+n-k\} \\ j & j < i \text{ or } j > i+n-k \end{cases}$$

for all $1 \leqslant i \leqslant k$. Then $E = \{g_1, \ldots, g_{k+1}, h_1, \ldots, h_k\}$ are all idempotents in D_{k+1} and they satisfy the hypothesis of the lemma. If $k = n - 2$, then $g_i = h_i$ for all $1 \leqslant i \leqslant k$, and so E contains $n - 1$ elements. If $k < n - 2$, then $|E| = 2k + 1$. From Lemma 2.3, we have that $|E| \geqslant n - 1$, as required. □

Proof of Theorem 1.1. Let $e_1, \ldots, e_{n-1}$ be the idempotents from Lemma 2.4. Note that $f \cdot g\phi = f$ and so $\operatorname{im}(f) \subseteq \operatorname{im}(g\phi)$ for all $g \in O_n$. In particular, $\operatorname{im}(f) \subseteq \operatorname{im}(e_i\phi)$ for all $1 \leqslant i \leqslant n - 1$. If $i \neq j$, then $e_ie_j \in I_k$ or $e_je_i \in I_k$. Hence $e_i\phi \cdot e_j\phi = f$ or $e_j\phi \cdot e_i\phi = f$. Thus every element in $\operatorname{im}(e_i\phi) \cap \operatorname{im}(e_j\phi)$ is fixed by f (as $e_i\phi$ and $e_j\phi$ are idempotents). It follows that

$$\operatorname{im}(e_i\phi) \cap \operatorname{im}(e_j\phi) = \operatorname{im}(f),$$

for all $i \neq j$.

Let $E_i = \operatorname{im}(e_i\phi) \setminus \operatorname{im}(f)$ for all i. Then $E_1, \ldots, E_{n-1}$ are pairwise disjoint. Since $e_1\phi, \ldots, e_{n-1}\phi \in D_{k+1}\phi$ it follows from Lemma 2.2 that $|E_1| = \cdots = |E_{n-1}| \geqslant 1$. It follows that $|E_1 \cup \cdots \cup E_{n-1}| \geqslant n - 1$. Therefore $|E_i| = 1$ for all i and $|\operatorname{im}(f)| = 1$. In other words, $D_{k+1}\phi \subseteq D_2$ and, again, $\operatorname{im}(f) \subseteq \operatorname{im}(g\phi)$ for all $g \in D_{k+1}$. Hence

$D_{k+1}\phi$ contains at most $n-1$ different $\mathcal{L}$-classes, and so, by Lemma 2.1, D_{k+1} has at most $n-1$ different $\mathcal{L}$-classes. Thus $k+1=n$, a contradiction, and so every endomorphism of O_n is of type (a), (b), or (c). $\square$

Proof of Theorem 1.2. We must prove that

$$|\operatorname{End}(O_n)| = 2 + \sum_{i=0}^{n-1} \binom{n+i}{2i+1} F_{2i+2}.$$

If X is a subset of O_n, then denote by $E(X)$ the set of all idempotents in X. It was shown in [7, Corollary 4.4] and [5, Theorem 2.3] that

$$|E(D_{i+1})| = \binom{n+i}{2i+1} \tag{1}$$

and

$$|E(O_n)| = F_{2n} \tag{2}$$

where F_{2n} is the $2n$ th Fibonacci number.

Let $e \in E(O_n)$ be arbitrary and let $S(e) = \{\, f \in E(O_n) \ :\ ef = fe = f \,\}$. Then the numbers of endomorphisms of O_n of types (b) and (c) where $1_n\phi = e$ are $|S(e)|-1$ and 1, respectively.

Let $e \in D_{i+1}$ where $0 \leqslant i \leqslant n-1$. Then we will prove that $|S(e)| = F_{2i+2}$.

Let $O_{\mathrm{im}(e)}$ be the semigroup of order-preserving mappings on $\mathrm{im}(e)$ and let $\Psi : S(e) \longrightarrow O_{\mathrm{im}(e)}$ be defined so that $(f)\Psi$ is the restriction $f|_{\mathrm{im}(e)}$ of f to $\mathrm{im}(e)$. If $f \in S(e)$, then $fe = f$ and so $\mathrm{im}(f) \subseteq \mathrm{im}(e)$. Hence Ψ is well-defined. Moreover $f|_{\mathrm{im}(e)}$ fixes $\mathrm{im}(f)$ pointwise, and so $f|_{\mathrm{im}(e)} \in E(O_{\mathrm{im}(e)})$.

We will prove that Ψ is a bijection from $S(e)$ to $E(O_{\mathrm{im}(e)})$. If $f \in E(O_{\mathrm{im}(e)})$, then $e \cdot ef = ef$ and $ef \cdot e = ef$ as $\mathrm{im}(f) \subseteq \mathrm{im}(e)$. It follows that $ef \in S(e)$ and $(ef)\Psi = (ef)|_{\mathrm{im}(e)} = f|_{\mathrm{im}(e)} = f$. That is, $\mathrm{im}(\Psi) = E(O_{\mathrm{im}(e)})$.

If $f, g \in S(e)$ such that $f|_{\mathrm{im}(e)} = g|_{\mathrm{im}(e)}$, then $f = ef = e \cdot f|_{\mathrm{im}(e)} = e \cdot g|_{\mathrm{im}(e)} = eg = g$, and so Ψ is injective. So, $|S(e)| = |E(O_{\mathrm{im}(e)})|$ and therefore we see from (2) that

$$|E(O_{\mathrm{im}(e)})| = F_{2|\mathrm{im}(e)|} = F_{2i+2}. \tag{3}$$

Therefore there are

$$|E(D_{i+1})|(|S(e)|-1) = \binom{n+i}{2i+1}(F_{2i+2}-1) \tag{4}$$

endomorphisms of type (b) where $1_n\phi \in D_{i+1}$.

There are two automorphisms and F_{2n} constant endomorphisms. Summing these two values and (4) over all i we obtain the required value. $\square$

Proof of Theorem 1.3. To prove the first assertion of the theorem, it suffices to prove that the $\mathcal{L}$-classes of $\operatorname{End}(O_n)$, except $\{1_n, \alpha\}$, are trivial. This follows from the fact that $\phi \cdot \phi_e = \phi_e$ for all $e \in E(O_n)$ and $\phi \in \operatorname{End}(O_n)$; that $\phi_h\phi_{e,f} = \phi_f$ for all appropriate $e, f, h \in E(O_n)$ such that $h \neq 1_n$; that $\phi_{1_n}\phi_{e,f} = \phi_e$ for all appropriate $e, f \in E(O_n)$; that $\phi_{i,j}\phi_{e,f} = \phi_f$ for all appropriate $e, f, i, j \in E(O_n)$ such that $i \neq 1_n$; and that $\phi_{1_n,h}\phi_{e,f} = \alpha\phi_{e,f} = \phi_{e,f}$ for all appropriate $e, f, h \in E(O_n)$.

To prove the second assertion, we start by noting that it is easy to prove that $\{\phi_e : e \in E(O_n)\}$ and $\mathrm{Aut}(O_n) = \{1_n, \alpha\}$ are $\mathcal{R}$-classes in $\mathrm{End}(O_n)$. Also note that the constant endomorphisms $\{\phi_e : e \in E(O_n)\}$ are the minimal ideal of $\mathrm{End}(O_n)$.

Furthermore, $\phi_{1_n,e}\phi_{1_n,f} = \phi_{1_n,f}$ for all $e, f \in E(O_n)$ such that $e, f \neq 1_n$. Moreover, every element $\phi_{e,f} \cdot \phi$ where $\phi \in \mathrm{End}(O_n) \setminus \{1_n, \alpha\}$, with $fe = fe = f$, $f \neq e$ and $e \neq 1_n$, is equal to ϕ_h for some $h \in E(O_n)$. Hence $\{\phi_{1_n,e} : e \in E(O_n),\ e \neq 1_n\}$ is an $\mathcal{R}$-class of $\mathrm{End}(O_n)$.

So, we are left to understand how the set $\{\phi_{e,f} : e, f \in E(O_n),\ fe = ef = f,\ f \neq e,\ e \neq 1_n\}$ splits into $\mathcal{R}$-classes. From the comments in the previous paragraph it follows that the $\mathcal{R}$-class of the element $\phi_{e,f}$ with $fe = ef = f$, $f \neq e$ and $e \neq 1_n$, is equal to $\{\phi_{e,f}, \phi_{e,f}\alpha\}$. This proves the theorem. □

Proof of Theorem 1.4. Since the $\mathcal{R}$-class $\{\phi_{e,f}, \phi_{e,f}\alpha\}$ where $fe = ef = f$, $f \neq e$ and $e \neq 1_n$ is a singleton if and only if $\phi_{e,f} = \phi_{e,f}\alpha$, it follows that the number of singleton $\mathcal{R}$-classes equals:

$$|\{\, (e,f) \in E(O_n) \times E(O_n) \ :\ fe = ef = f,\ f \neq e,\ e \neq 1_n,\ (e)\alpha = e,\ (f)\alpha = f \,\}|.$$

There are two cases to consider.

Case (i). $n = 2k$. Let $e, f \in E(O_n)$ such that $ef = fe = f$, $f \neq e$, $e \neq 1_n$, $(e)\alpha = e$ and $(f)\alpha = f$. Recall that α is the non-identity automorphism of O_n and $(s)\alpha = s^\sigma$ for all $s \in O_n$, where $\sigma = (1\,n)(2\,n-1)\cdots(\lfloor n/2\rfloor\ \lceil n/2\rceil + 1)$. If $i \in \{1, \ldots, k\}$, then

$$(2k+1-i)e = (2k+1-i)(e\alpha) = (2k+1-i)(e^\sigma) = 2k+1-(i)e \tag{5}$$

and analogously $(2k+1-i)f = 2k+1-(i)f$. Hence

$$(i)e \leqslant (2k+1-i)e = 2k+1-(i)e$$

for all $i \leqslant k$. Therefore $(i)e \leqslant k$. This means that $e|_{\{1,\ldots,k\}}$ is a function from $\{1, \ldots, k\}$ to $\{1, \ldots, k\}$. Analogously one proves that $f|_{\{1,\ldots,k\}}$ is a function from $\{1, \ldots, k\}$ to $\{1, \ldots, k\}$; and that $e|_{\{k+1,\ldots,2k\}}$ and $f|_{\{k+1,\ldots,2k\}}$ are functions from $\{k+1, \ldots, 2k\}$ to $\{k+1, \ldots, 2k\}$. Furthermore, from (5) it follows that $e|_{\{k+1,\ldots,2k\}}$ is a 'mirror-reflection' of $e|_{\{1,\ldots,k\}}$ with respect to the middlepoint of the interval $[1, 2k]$; and the same holds if we replace e by f. Since e and f are idempotents, and their restrictions to $\{1, \ldots, k\}$ and $\{k+1, \ldots, 2k\}$ act within their domains, we obtain that $e|_{\{1,\ldots,k\}}$ and $f|_{\{1,\ldots,k\}}$ are idempotents from $O_{\{1,\ldots,k\}}$. Moreover, since $ef = fe = f$, we have that $e|_{\{1,\ldots,k\}}f|_{\{1,\ldots,k\}} = f|_{\{1,\ldots,k\}}e|_{\{1,\ldots,k\}} = f|_{\{1,\ldots,k\}}$. In notation from the proof of Theorem 1.2 this translates as $f|_{\{1,\ldots,k\}} \in S(e|_{\{1,\ldots,k\}})$. Our final note is that $e \neq f$ implies $e|_{\{1,\ldots,k\}} \neq f|_{\{1,\ldots,k\}}$; and $e \neq 1_n$ implies $e|_{\{1,\ldots,k\}} \neq 1_k$.

To summarize what we discussed in the previous paragraph: e and f are completely determined by their restrictions to $\{1, \ldots, k\}$ which are idempotents x and y from $O_k = O_{\{1,\ldots,k\}}$ such that $y \in S(x)$, $x \neq 1_k$ and $x \neq y$. Vice versa, for any $x, y \in E(O_k)$ such that $y \in S(x)$ and $x \neq y$ there exist unique $e, f \in E(O_n)$ such that $ef = fe = f$, $f \neq e$, $e \neq 1_n$, $(e)\alpha = e$, $(f)\alpha = f$, $e|_{\{1,\ldots,k\}} = x$ and $f|_{\{1,\ldots,k\}} = y$. Also recall from (3) that for every $x \in O_k$, $|S(x)| = F_{2|\mathrm{im}(x)|}$. Hence the required number is equal to

$$\sum_{x \in E(O_k)\setminus\{1_k\}} (|S(x)| - 1) = -(F_{2k} - 1) + \sum_{x \in E(O_k)} (F_{2|\mathrm{im}(x)|} - 1).$$

Let $x \in E(O_k)$ and let rank$(x) = i+1$ for some $0 \leqslant i \leqslant k-1$. By (1), the number of idempotents in O_k of rank $i+1$ is equal to $\binom{k+i}{2i+1}$. Therefore the required number is equal to

$$-(F_{2k}-1) + \sum_{x \in E(O_k)} (F_{2|\mathrm{im}(x)|}-1) = \sum_{i=0}^{k-1} (F_{2i+2}-1)\binom{k+i}{2i+1} - (F_{2k}-1).$$

Case (ii). $n = 2k+1$. By the same token as in Case (i), there is a one-one correspondence between pairs of idempotents $e, f \in E(O_n)$ such that $ef = fe = f$, $f \neq e$, $e \neq 1_n$, $(e)\alpha = e$, $(f)\alpha = f$, and pairs of idempotents $x, y \in O_{k+1}$ such that $y \in S(x)$, $x \neq 1_{k+1}$, $x \neq y$ and $(k+1)x = (k+1)y = k+1$. Hence the required number is equal to

$$\sum_{x \in E(O_{k+1}) \setminus \{1_{k+1}\}, (k+1)x = k+1} (|S(x)|-1) =$$
$$- (F_{2(k+1)}-1) + \sum_{x \in E(O_{k+1}), (k+1)x=k+1} (|S(x)|-1).$$

We will now calculate the latter sum, associating to each $x \in E(O_{k+1})$ with $(k+1)x = k+1$ two numbers i and j. So, take $x \in E(O_{k+1})$ with $(k+1)x = k+1$. Let $|(k+1)x^{-1}| = i+1$ for some $0 \leqslant i \leqslant k$. Then $(k+1)x^{-1} = \{k-i+1, \ldots, k+1\}$. Let also rank$(x) = j$. Then j can lie in the range from 1 to $k-i+1$. Vice versa, fixing i and j with $0 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k-i+1$, the number of idempotents $x \in E(O_{k+1})$ such that $(k+1)x = k+1$ with these correspondent numbers i and j, is equal to the number of idempotents from $O_{\{1,\ldots,k-i\}}$ of rank $j-1$, which is $\binom{k-i+j-2}{2j-3}$ by (1). Moreover, for each such x, $|S(x)| = F_{2j}$. Therefore the required number is equal to

$$- (F_{2(k+1)}-1) + \sum_{x \in E(O_{k+1}), (k+1)x=k+1} (|S(x)|-1) =$$
$$\sum_{i=0}^{k} \sum_{j=1}^{k-i+1} (F_{2j}-1)\binom{k-i+j-2}{2j-3} - (F_{2(k+1)}-1),$$

as required. □

Proof of Corollary 1.5. Let us calculate the number of elements $\phi_{e,f}$ with $ef = fe = f$, $f \neq e$ and $e \neq 1_n$. The number of elements $\phi_{1_n,e}$ with $e \neq 1_n$ is equal to $F_{2n}-1$; and the number of elements ϕ_e is equal to F_{2n}. Hence by Theorem 1.2, we have that the number of elements $\phi_{e,f}$ with $ef = fe = f$, $f \neq e$ and $e \neq 1_n$ is equal to

$$\sum_{i=0}^{n-1} \binom{n+i}{2i+1} F_{2i+2} - (2F_{2n}-1).$$

The result now follows immediately from Theorems 1.2, 1.3 and 1.4. □

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