RELATIVE RANKS OF LIPSCHITZ MAPPINGS ON COUNTABLE DISCRETE METRIC SPACES

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Abstract. Let $\mathfrak{X}$ be a countable discrete metric space and let $\mathfrak{X}^{\mathfrak{X}}$ denote the family of all functions on $\mathfrak{X}$. In this article, we consider the problem of finding the least cardinality of a subset A of $\mathfrak{X}^{\mathfrak{X}}$ such that every element of $\mathfrak{X}^{\mathfrak{X}}$ is a finite composition of elements of A and Lipschitz functions on $\mathfrak{X}$. It follows from a classical theorem of Sierpiński that such an A either has size at most 2 or is uncountable.

We show that if $\mathfrak{X}$ contains a Cauchy sequence or a sufficiently separated, in some sense, subspace, then $|A| \leqslant 1$. On the other hand, we give several results relating $|A|$ to the cardinal $\mathfrak{d}$; defined as the minimum cardinality of a dominating family for $\mathbb{N}^{\mathbb{N}}$. In particular, we give a condition on the metric of $\mathfrak{X}$ under which $|A| \geqslant \mathfrak{d}$ holds and a further condition that implies $|A| \leqslant \mathfrak{d}$. Examples satisfying both of these conditions include all subsets of $\mathbb{N}^k$ and the sequence of partial sums of the harmonic series with the usual euclidean metric.

To conclude, we show that if $\mathfrak{X}$ is any countable discrete subset of the real numbers $\mathbb{R}$ with the usual euclidean metric, then $|A| = 1$ or almost always, in the sense of Baire category, $|A| = \mathfrak{d}$.

1. Introduction and preliminaries

Let X be a semigroup and let Y be a subset of X. The *relative rank* of X modulo Y, denoted by $\operatorname{rank}(X : Y)$, is defined as the least cardinality of a subset Z of X such that $Y \cup Z$ generates X. We may also refer to $\operatorname{rank}(X : Y)$ as the relative rank of Y in X.

The subject of relative ranks of subsemigroups of transformation semigroups has been intensively investigated. The roots of the study can be traced back to Sierpiński [16] and Banach [2]; the notion of relative ranks first appeared explicitly in [12] and [13]. The ranks of several standard examples of subsemigroups, such as the symmetric group, the set of idempotents and so on, of the full transformation semigroup and related semigroups were determined in [10], [12], and [13]. In [1], [9], and [11] similar considerations we made for other classes of semigroups, such as order preserving mappings and linear mappings of vector spaces. Galvin [8], and Bergman and Shelah [4] considered related topics for permutation groups. The relative rank of Lebesgue measurable subgroups of the reals under addition were also studied, under a different guise, in [5].

The topic of relative ranks of continuous mappings $\mathcal{C}_{\mathfrak{X}}$ modulo Lipschitz mappings $\mathfrak{L}_{\mathfrak{X}}$ on metric spaces $\mathfrak{X}$ was considered in [6]. It was shown that for two large classes of metric spaces, which include many natural examples, $\operatorname{rank}(\mathcal{C}_{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}})$

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is uncountable. Moreover, it was shown that the relative rank of the semigroup of Lipschitz mappings $\mathfrak{L}_{\mathcal{N}}$ on the Baire space $\mathcal{N}$ in the semigroup of all continuous mappings $\mathcal{C}_{\mathcal{N}}$ on $\mathcal{N}$ is equal to the smallest uncountable cardinal $\aleph_1$, i.e. $\operatorname{rank}(\mathcal{C}_{\mathcal{N}} : \mathfrak{L}_{\mathcal{N}}) = \aleph_1$. Somewhat surprisingly, if an arbitrary point x is removed from the Baire space $\mathcal{N}$, then this rank becomes 1.

The example of the Baire space illustrates that the cardinal $\operatorname{rank}(\mathcal{C}_{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}})$ is sensitive to changes in the metric structure of $\mathfrak{X}$ even if its topological structure remains the same; the spaces $\mathcal{N}$ and $\mathcal{N}\setminus\{x\}$ are homeomorphic. Although all countable discrete metric spaces $\mathfrak{X}$ are topologically identical, their metric structures can vary widely, and it is natural to ask how the differences in these metrics affect $\operatorname{rank}(\mathcal{C}_{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}})$. Moreover, in this particular case the semigroup of all continuous mappings on $\mathfrak{X}$ coincides with the well investigated semigroup $\mathfrak{X}^{\mathfrak{X}}$ of all mappings from $\mathfrak{X}$ to $\mathfrak{X}$. Hence, by Sierpiński's Theorem [16], $\operatorname{rank}(\mathcal{C}_{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \in \{0, 1, 2\}$ or $> \aleph_0$.

Of course, when $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) > \aleph_0$, if we assume that the Continuum Hypothesis (**CH**) holds, then $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = 2^{\aleph_0}$, the cardinality of the continuum. However, if we do not assume that **CH** holds, then the question of the precise value of $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}})$ remains open. This question motivated the majority of the results presented here.

To specify the cardinal numbers involved we require the following notions. A function $f \in \mathbb{N}^{\mathbb{N}}$ is said to *dominate* another function $g \in \mathbb{N}^{\mathbb{N}}$ if $f(i) \geqslant g(i)$ for all $i \in \mathbb{N}$. The study of the notion of dominance, and related ideas, in the 70s and 80s of the previous century gave rise to the following cardinal numbers, introduced by van Douwen. The cardinal $\mathfrak{b}$ is the least cardinality of a subset U of $\mathbb{N}^{\mathbb{N}}$ such that for any countable $V \subseteq \mathbb{N}^{\mathbb{N}}$ there exists $f \in U$ such that f is not dominated by any $g \in V$. The cardinal $\mathfrak{d}$ is defined to be the least cardinal of a family $\mathcal{F}$ such that for all $f \in \mathbb{N}^{\mathbb{N}}$ there exists $g \in \mathcal{F}$ that dominates f. Cichoń's diagram connects the cardinals $\mathfrak{b}$ and $\mathfrak{d}$, and as such the notion of dominance in $\mathbb{N}^{\mathbb{N}}$, with other cardinals related to Baire category and Lebesgue measure; see [3] or [7].

The following relations are obvious: $\aleph_1 \leqslant \mathfrak{b} \leqslant \mathfrak{d} \leqslant 2^{\aleph_0}$. If the **CH** holds, then $\mathfrak{b} = \mathfrak{d} = 2^{\aleph_0}$. Moreover, the following theories are consistent: ZFC + $(\mathfrak{b} = \mathfrak{d} = \aleph_1 < 2^{\aleph_0})$, ZFC + $(\aleph_1 < \mathfrak{b} = \mathfrak{d} = \aleph_2 = 2^{\aleph_0})$ and ZFC + $(\aleph_1 = \mathfrak{b} < \mathfrak{d} = 2^{\aleph_0})$; see [3].

Let $\mathfrak{X}$ be a countable discrete metric space. In Section 2, we give several sufficient conditions for $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}})$ to be finite. For example, if $\mathfrak{X}$ contains a Cauchy sequence, then $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = 1$. The main theorem of Section 3 asserts that if every open ball in $\mathfrak{X}$ is finite, then $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}})$ is at least $\mathfrak{d}$. A condition for $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}})$ to be at most $\mathfrak{d}$ is given in the main theorem of Section 4. Several natural metric spaces $\mathfrak{X}$ are shown to satisfy the hypotheses of both theorems and thus have $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$. For instance, any infinite subset $\mathfrak{X}$ of $\mathbb{N}^k$ for any $k \geqslant 1$ satisfies $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$. In Section 5 we deduce from the more general theorems of the preceding sections that the countable subsets $\mathfrak{X}$ of $\mathbb{R}$ satisfy either $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = 1$ or $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \geqslant \mathfrak{d}$. In Theorem 5.4 we show that $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$ for a large class of countable subsets of $\mathbb{R}$. In fact, in Section 6, we prove that almost all, in the sense of Baire category, countable subsets $\mathfrak{X}$ of $\mathbb{R}$ have $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$.

As usual, if S is a semigroup and $U \subseteq S$, then we denote by $\langle U \rangle$ the subsemigroup generated by U. For brevity, if $U, V \subseteq S$, then we will write $\langle\, U, V\,\rangle$ instead of the formally correct $\langle\, U \cup V\,\rangle$.

2. Spaces with finite rank

Throughout this section let $\mathfrak{X}$ denote a countable discrete metric space with metric d. We now give sufficient conditions on d for $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}})$ to be finite. We will say that a metric d is *bounded below* to mean there exists $\epsilon > 0$ such that $d(x, y) > \epsilon$ for all $x \neq y$.

Theorem 2.1. *The metric d is bounded above and below on the entire space $\mathfrak{X}$ if and only if* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = 0$.

Proof. If α is a lower bound on $d(x, y)$ for all $x, y \in \mathfrak{X}$ and β an upper bound, then every mapping in $\mathfrak{X}^{\mathfrak{X}}$ is Lipschitz with constant β/α. Hence $\mathfrak{L}_{\mathfrak{X}} = \mathfrak{X}^{\mathfrak{X}}$ and $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = 0$.

In either of the cases that d is not bounded above or d is not bounded below on $\mathfrak{X}$, there is a mapping that is not Lipschitz. Thus $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) > 0$. □

For subsets $U, V \subseteq \mathfrak{X}$ let $\overline{d}(U, V) = \inf\{d(u, v) : u \in U, v \in V\}$. Throughout the paper, if A is any set and we write $A = \{a_1, a_2, \ldots\}$, then we assume that $a_i \neq a_j$ whenever $i \neq j$. Furthermore, if A is finite, then we may write $A = \{a_1, a_2, \ldots\}$ to mean $A = \{a_1, a_2, \ldots, a_n\}$.

Theorem 2.2. *Let $\mathfrak{X}$ contain an infinite subset U such that $\overline{d}(U, \mathfrak{X} \setminus U) > 0$ or $\mathfrak{X} = U$, d is bounded above on U, and U can be partitioned into countably many non-empty sets $U_1, U_2, \ldots$ satisfying* $\inf_{i,j} \overline{d}(U_i, U_j) > 0$. *Then* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \leqslant 1$.

Proof. Let $U = \{u_1, u_2, \ldots\}$ and let $V = \{v_1, v_2, \ldots\}$ be any set where $v_i \in U_i$ for all i. The metric d is bounded above and below on V by positive reals β and $\alpha = \inf_{i,j} \overline{d}(U_i, U_j) (> 0)$, respectively. Every mapping from V to U is Lipschitz with constant $\lambda = \beta/\alpha$. Furthermore, every mapping $l : V \longrightarrow U$ can be extended to a Lipschitz mapping $\widehat{l} : \mathfrak{X} \longrightarrow \mathfrak{X}$, by fixing all the elements in $\mathfrak{X} \setminus U$ pointwise and mapping every $x \in U_i$ to $l(v_i)$. In fact, if $u, v \in \mathfrak{X} \setminus U$, then obviously $\widehat{l}$ satisfies $d(\widehat{l}(u), \widehat{l}(v)) = d(u, v)$. Let $u \in U_i$ and $v \in U_j$ for some i, j. If $i = j$, then $d(\widehat{l}(u), \widehat{l}(v)) = 0$. Otherwise, $d(\widehat{l}(u), \widehat{l}(v)) = d(\widehat{l}(v_i), \widehat{l}(v_j)) \leqslant (\beta/\alpha) d(u, v)$. Finally, if $u \in U_i$ and $v \in \mathfrak{X} \setminus U$, then

$$d(\widehat{l}(u), \widehat{l}(v)) = d(\widehat{l}(v_i), v) \leqslant d(\widehat{l}(v_i), u) + d(u, v) \leqslant \beta + d(u, v) \leqslant (\beta/\gamma + 1) d(u, v)$$

where $\gamma = \overline{d}(U, \mathfrak{X} \setminus U) > 0$.

To complete the proof, it is necessary to define a mapping $f : \mathfrak{X} \longrightarrow \mathfrak{X}$ such that every element of $\mathfrak{X}^{\mathfrak{X}}$ can be given as a finite composition of elements of $\mathfrak{L}_{\mathfrak{X}}$ and f. Partition V into sets $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$, and C such that $|A| = |C| = |U|$ and $|B| = |\mathfrak{X} \setminus U|$. Also let $\mathfrak{X} \setminus U = \{d_1, d_2, \ldots\}$. Note that the sets B and $\mathfrak{X} \setminus U$ can be finite, even empty, or infinite.

Let $t : (U \setminus V) \cup C \longrightarrow C$ be a bijection. It is now possible to define the mapping f as follows (if $\mathfrak{X} = U$, then the second and third clauses in the definition of f are vacuous)

$$f(x) = \begin{cases} x & x \in A \\ d_i & x = b_i \\ b_i & x = d_i \\ t(x) & x \in (U \setminus V) \cup C. \end{cases}$$

Note that f maps $\mathfrak{X} \setminus U$ bijectively to B, and vice versa, and $\operatorname{im}(f) = (\mathfrak{X} \setminus U) \cup V$.

Before doing anything else, we prove that there exists a bijection from $\mathfrak{X}$ to $A \cup B$ that is a composition of f and an element of $\mathfrak{L}_{\mathfrak{X}}$. If $\widehat{l_1}$ is the extension, as defined above, of any bijection $l_1 : V \longrightarrow A$ to a Lipschitz mapping, then

$$h = f \circ \widehat{l_1} \circ f : \mathfrak{X} \longrightarrow A \cup B$$

is the desired bijection.

Let $g \in \mathfrak{X}^{\mathfrak{X}}$ be an arbitrary element. Then in our decomposition of g there must be one Lipschitz mapping that depends on g. Let $l_2 : V \longrightarrow A \cup B$ be any function such that for all $y \in A \cup B$ we have that

$$l_2(y) = \begin{cases} a_i & \text{if } g(h^{-1}(y)) = u_i \\ b_i & \text{if } g(h^{-1}(y)) = d_i. \end{cases}$$

Then l_2 can be extended to a Lipschitz mapping $\widehat{l_2}$ as described above.

The final Lipschitz mapping that we require is the extension $\widehat{l_3}$ of any mapping $l_3 : V \longrightarrow U$ such that $l_3(a_i) = u_i$ for all $a_i \in A$.

So, if $x \in \mathfrak{X}$ and $g(x) = u_i \in U$, then

$$\widehat{l_3} \circ f \circ \widehat{l_2} \circ f \circ \widehat{l_1} \circ f(x) = \widehat{l_3} \circ f \circ \widehat{l_2}(h(x)) = \widehat{l_3} \circ f(a_i) = \widehat{l_3}(a_i) = u_i = g(x).$$

Finally, if $x \in \mathfrak{X}$ and $g(x) = d_i \in \mathfrak{X} \setminus U$, then

$$\widehat{l_3} \circ f \circ \widehat{l_2} \circ f \circ \widehat{l_1} \circ f(x) = \widehat{l_3} \circ f \circ \widehat{l_2}(h(x)) = \widehat{l_3} \circ f(b_i) = \widehat{l_3}(d_i) = d_i = g(x).$$

Hence $\mathfrak{X}^{\mathfrak{X}} = \langle\, \mathfrak{L}_{\mathfrak{X}}, f \,\rangle$. □

The following corollaries are immediate consequences of Theorem 2.2.

Corollary 2.3. *If $\mathfrak{X}$ contains an infinite subset U such that $\overline{d}(U, \mathfrak{X} \setminus U) > 0$ or $\mathfrak{X} = U$ and d is bounded above and below on U, then* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \leqslant 1$.

Corollary 2.4. *If d is bounded below but not above and $\mathfrak{X}$ contains an infinite subset where d is bounded above, then* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = 1$.

The next theorem provides another way of determining the relative rank of $\mathfrak{L}_{\mathfrak{X}}$ in $\mathfrak{X}^{\mathfrak{X}}$ in certain cases.

Theorem 2.5. *If $\mathfrak{X}$ contains a Cauchy sequence of distinct elements, then* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = 1$.

Proof. Since d is not bounded below, $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \geqslant 1$. The opposite inequality is proved using an argument similar to that used in the proof of Theorem 2.2.

Let $\mathfrak{X} = \{x_1, x_2, \ldots\}$ and let $U = (u_1, u_2, \ldots)$ be a Cauchy sequence of distinct elements of $\mathfrak{X}$. Assume without loss of generality that $d(u_m, u_n) < 1/\min\{m, n\}$. Let $x \in \mathfrak{X}$ be arbitrary and define

$$\alpha(x) = \inf\{\, d(x, y) \;:\; y \in \mathfrak{X}, y \neq x \,\}.$$

Note that $\alpha(x) > 0$ for all $x \in \mathfrak{X}$ since $\mathfrak{X}$ is discrete. Partition U into infinitely many infinite sets $U_1, U_2, \ldots$ and let f be any function in $\mathfrak{X}^{\mathfrak{X}}$ that is constant with value x_i everywhere on U_i.

Let $g \in \mathfrak{X}^{\mathfrak{X}}$ be arbitrary. We define a function $l : \mathfrak{X} \longrightarrow U$ (that in some sense encodes g) recursively, starting with $l(x_1) = u_{n(1)} \in U_{m(1)}$ where $g(x_1) = x_{m(1)}$ and $n(1) > 1/\alpha(x_1)$. Thereafter, for $i > 1$, define $l(x_i) = u_{n(i)} \in U_{m(i)}$ where $g(x_i) = x_{m(i)}$ and $n(i) > \max\{n(1), \ldots, n(i-1), 1/\alpha(x_i)\}$.

If $x_i, x_j \in \mathfrak{X}$ are arbitrary, then

$$\begin{aligned} d(l(x_i), l(x_j)) = d(u_{n(i)}, u_{n(j)}) < 1/\min\{n(i), n(j)\} &= \max\{1/n(i), 1/n(j)\} \\ &\leqslant \max\{\alpha(x_i), \alpha(x_j)\} \leqslant d(x_i, x_j). \end{aligned}$$

Thus l is Lipschitz with constant 1.

To conclude, if $x_i \in \mathfrak{X}$, then $f \circ l(x_i) = f(u_{n(i)}) = x_{m(i)} = g(x_i)$. □

The next example is that of a space that satisfies the condition of Theorem 2.2 but not that of Theorem 2.5.

Example 2.6. Let $\mathfrak{X} = \{x_1, x_2, \ldots\}$ (recall that we use the convention that $x_i \neq x_j$ if $i \neq j$). Define a metric d on $\mathfrak{X}$ by

$$d(x_i, x_j) = \begin{cases} 0 & i = j \\ \frac{1}{k} & \{i, j\} = \{2k-1, 2k\} \text{ for some } k \in \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, $\mathfrak{X}$ contains no Cauchy sequences and d is bounded above by 1 on $\mathfrak{X}$. Although the metric is unbounded below, if $U_k = \{x_{2k-1}, x_{2k}\}$, for all $k \in \mathbb{N}$, then $\inf_{i,j} \overline{d}(U_i, U_j) = 1$. Thus by Theorems 2.1 and 2.2, $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = 1$.

3. AT LEAST $\mathfrak{d}$

In this section we consider those metric spaces $\mathfrak{X}$ where the open balls $B(x, r) = \{\, y \in \mathfrak{X} \;:\; d(x, y) < r \,\}$ are finite for all $x \in \mathfrak{X}$ and for all $r > 0$. Note that the condition that all balls are finite is equivalent to d being unbounded above on every infinite subset of $\mathfrak{X}$. In order to provide succinct proofs of the theorems in this section we require some auxiliary notions taken from Bergman and Shelah [4] and Mesyan [15].

Let U, V be subsets of $\mathfrak{X}^{\mathfrak{X}}$. Then we will write $U \preccurlyeq V$ if there exists a countable subset C of $\mathfrak{X}^{\mathfrak{X}}$ such that $U \subseteq \langle V, C \rangle$. We will write $U \approx V$ if $U \preccurlyeq V$ and $V \preccurlyeq U$. It is straightforward to show that $\preccurlyeq$ is a preorder and that $\approx$ is an equivalence relation on the subsets of $\mathfrak{X}^{\mathfrak{X}}$. Of course, if $U \preccurlyeq V$ and $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : V)$ is infinite (and hence uncountable by Sierpiński's Theorem [16]), then $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : U) \geqslant \operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : V)$.

Let $\mathfrak{X} = \{x_1, x_2, \ldots\}$ and let $\mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ denote the semigroup of functions f in $\mathfrak{X}^{\mathfrak{X}}$ satisfying $f(x_i) \in \{x_1, x_2, \ldots, x_i\}$ for all $x_i \in \mathfrak{X}$. Of course, the definition of $\mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ depends on the enumeration $x_1, x_2, \ldots$ of $\mathfrak{X}$. However, it is straightforward to verify that any two enumerations give rise to semigroups that are equivalent under $\approx$ and so for the problems considered here we can write $\mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ without ambiguity. When $\mathfrak{X} = \mathbb{N}$ we assume that $f(i) \leqslant i$ for all $f \in \mathbb{N}^{\mathbb{N}}_{\leqslant}$ and $i \in \mathbb{N}$.

We will prove a series of lemmas that lead to the proof of the following theorem.

Theorem 3.1. *If every open ball in $\mathfrak{X}$ is finite, then $\mathfrak{L}_{\mathfrak{X}} \preccurlyeq \mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ and* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \geqslant \mathfrak{d}$.

Recall that a function $f \in \mathbb{N}^{\mathbb{N}}$ is said to *dominate* another function $g \in \mathbb{N}^{\mathbb{N}}$ if $f(i) \geqslant g(i)$ for all $i \in \mathbb{N}$. A subset U of $\mathbb{N}^{\mathbb{N}}$ *dominates a family* $V \subseteq \mathbb{N}^{\mathbb{N}}$ if for all $g \in V$ there exists $f \in U$ such that f dominates g. Let $f \in U$ be arbitrary and define $f' \in \mathbb{N}^{\mathbb{N}}$ inductively by

$$f'(i) = \max(f'(i-1) + 1, f(i)).$$

In this way it is possible to replace any dominating family U for V by a dominating family U' that consists entirely of strictly increasing functions and where $|U'| = |U|$.

The following lemma is routine and the proof is omitted.

Lemma 3.2. *Let U, V be arbitrary subsets of $\mathbb{N}^{\mathbb{N}}$ and let X, Y be dominating families for U, V, respectively, consisting of increasing mappings. Then $\langle\, U, V \,\rangle$ is dominated by $\langle\, X, Y \,\rangle$.*

A function $f \in \mathbb{N}^{\mathbb{N}}$ is said to be *eventually dominated* by another function $g \in \mathbb{N}^{\mathbb{N}}$ if there exists $m \in \mathbb{N}$ such that $f(n) \geqslant g(n)$ for all $n \geqslant m$. A function g eventually dominates a family of mappings U in $\mathbb{N}^{\mathbb{N}}$ if it eventually dominates every element of U.

Lemma 3.3. *Let U be an arbitrary subset of $\mathbb{N}^{\mathbb{N}}$. Then the following are equivalent:*

(i) *U is eventually dominated by a single mapping;*
(ii) *U is dominated by a countable family;*
(iii) *$U \preccurlyeq \mathbb{N}^{\mathbb{N}}_{\leqslant}$.*

Proof. (i) $\iff$ (ii) is straightforward to verify.

(ii)$\Rightarrow$(iii) Let V denote a countable dominating family for U. As mentioned above, we may assume that V consists of strictly increasing, and hence injective, functions. Now, if $f \in U$ is arbitrary, then let $g \in V$ denote any function that dominates f. Then there exists $h \in \mathbb{N}^{\mathbb{N}}_{\leqslant}$ such that $h(g(i)) = f(i)$ for all $i \in \mathbb{N}$, since $g(i) \geqslant f(i)$ for all $i \in \mathbb{N}$. Hence $f = h \circ g$ and so $U \subseteq \langle\, \mathbb{N}^{\mathbb{N}}_{\leqslant}, V \,\rangle$.

(iii)$\Rightarrow$(ii) Since $U \preccurlyeq \mathbb{N}^{\mathbb{N}}_{\leqslant}$ there exist a countable set $C \subseteq \mathbb{N}^{\mathbb{N}}$ such that $U \subseteq \langle\, \mathbb{N}^{\mathbb{N}}_{\leqslant}, C \,\rangle$. Thus $\langle\, \mathbb{N}^{\mathbb{N}}_{\leqslant}, C \,\rangle$ dominates U. The monoid $\mathbb{N}^{\mathbb{N}}_{\leqslant}$ is dominated by the identity mapping $1_{\mathbb{N}}$. Hence, by Lemma 3.2, U is dominated by $\langle\, C, 1_{\mathbb{N}} \,\rangle$, which is countable. □

Recall that the cardinal $\mathfrak{b}$ is the least cardinal of a family of mappings in $\mathbb{N}^{\mathbb{N}}$ that is not eventually dominated by any single element of $\mathbb{N}^{\mathbb{N}}$.

Corollary 3.4. *Let U be any subset of $\mathbb{N}^{\mathbb{N}}$ with $|U| < \mathfrak{b}$. Then $U \preccurlyeq \mathbb{N}^{\mathbb{N}}_{\leqslant}$ and $U \not\approx \mathbb{N}^{\mathbb{N}}_{\leqslant}$.*

Lemma 3.5. *The relative rank of $\mathbb{N}^{\mathbb{N}}_{\leqslant}$ in $\mathbb{N}^{\mathbb{N}}$ is $\mathfrak{d}$. Likewise, $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{X}^{\mathfrak{X}}_{\leqslant}) = \mathfrak{d}$.*

Proof. We start by proving that $\operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathbb{N}^{\mathbb{N}}_{\leqslant}) \leqslant \mathfrak{d}$. Let U be any dominating family for $\mathbb{N}^{\mathbb{N}}$ with cardinality $\mathfrak{d}$. Then $\langle\, \mathbb{N}^{\mathbb{N}}_{\leqslant}, U \,\rangle = \mathbb{N}^{\mathbb{N}}$ by the same argument as that used in the proof that (ii) implies (iii) in Lemma 3.3.

To prove that $\operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathbb{N}^{\mathbb{N}}_{\leqslant}) \geqslant \mathfrak{d}$ note that if $\langle\, \mathbb{N}^{\mathbb{N}}_{\leqslant}, U \,\rangle = \mathbb{N}^{\mathbb{N}}$ for some U, then $\langle\, \mathbb{N}^{\mathbb{N}}_{\leqslant}, U \,\rangle$ dominates $\mathbb{N}^{\mathbb{N}}$. Let V be a family of increasing mappings that dominates U such that $|V|$ is at most $|U|$. Then, by Lemma 3.2, $\langle\, \mathbb{N}^{\mathbb{N}}_{\leqslant}, U \,\rangle$ is dominated by $\langle\, 1_{\mathbb{N}}, V \,\rangle$. Hence $\langle\, 1_{\mathbb{N}}, V \,\rangle$ dominates $\mathbb{N}^{\mathbb{N}}$ and so $\mathfrak{d} \leqslant |\langle\, 1_{\mathbb{N}}, V \,\rangle| = |V| \leqslant |U|$. Thus $\operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathbb{N}^{\mathbb{N}}_{\leqslant}) \geqslant \mathfrak{d}$.

It remains to prove that $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{X}^{\mathfrak{X}}_{\leqslant}) = \mathfrak{d}$. Let $\phi : \mathfrak{X} \longrightarrow \mathbb{N}$ be any bijection. Then $\Phi : \mathfrak{X}^{\mathfrak{X}} \longrightarrow \mathbb{N}^{\mathbb{N}}$ defined by $\Phi(f)(x) = \phi \circ f \circ \phi^{-1}(x)$ is an isomorphism and $\Phi(\mathfrak{X}^{\mathfrak{X}}_{\leqslant}) \approx \mathbb{N}^{\mathbb{N}}_{\leqslant}$. Thus $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{X}^{\mathfrak{X}}_{\leqslant}) = \mathfrak{d}$. □

The following alternative proof that $\operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathbb{N}^{\mathbb{N}}_{\leqslant}) \geqslant \mathfrak{d}$ was suggested by S. Solecki. By Tychonoff's Theorem [14, Proposition 4.1(vi)], $\mathbb{N}^{\mathbb{N}}_{\leqslant}$ is compact in $\mathbb{N}^{\mathbb{N}}$

with the product topology. The mappings $\Phi_f, \Psi_f : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}$ defined by $\Phi_f(g) = f \circ g$ and $\Psi_f(g) = g \circ f$ are continuous on $\mathbb{N}^{\mathbb{N}}$. Using the fact that the continuous image of a compact set is compact, it follows that $\langle\, \mathbb{N}^{\mathbb{N}}_{\leqslant}, U \,\rangle$ is the union of $|U|$ compact sets. If A is a compact subset of $\mathbb{N}^{\mathbb{N}}$, then $\{\, f(i) \;:\; f \in A \,\}$ is finite for all $i \in \mathbb{N}$. Hence every compact set is dominated by a single mapping, and so $\mathbb{N}^{\mathbb{N}} = \langle\, \mathbb{N}^{\mathbb{N}}_{\leqslant}, U \,\rangle$ is dominated by $|U|$ mappings. Thus $|U| \geqslant \mathfrak{d}$.

Proof of Theorem 3.1. Assume without loss of generality that $\mathfrak{X} = \mathbb{N}$ with the metric d from the hypothesis of the theorem. Note that d is not necessarily the usual euclidean metric on $\mathbb{N}$. Let

$$L(c, i) = \{\, f \in \mathfrak{L}_{\mathbb{N}} \;:\; f \text{ is Lipschitz with constant } c \;\&\; f(1) = i \,\}.$$

Then for all $f \in L(c, i)$ and for all $n \in \mathbb{N}$ we have that $f(n) \in B(i, c \cdot d(n, 1))$. Define $h_{c,i} \in \mathbb{N}^{\mathbb{N}}$ by $h_{c,i}(n) = \max\{B(i, c \cdot d(n, 1))\}$. Clearly $h_{c,i}$ dominates $L(c, i)$ and so $\mathfrak{L}_{\mathbb{N}}$ is countably dominated. Hence from Lemma 3.3 we know that $\mathfrak{L}_{\mathbb{N}} \preccurlyeq \mathbb{N}^{\mathbb{N}}_{\leqslant}$ and it follows from Lemma 3.5 that $\operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathfrak{L}_{\mathbb{N}}) \geqslant \operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathbb{N}^{\mathbb{N}}_{\leqslant}) = \mathfrak{d}$. ☐

4. At most $\mathfrak{d}$

In this section we give some sufficient conditions on the space $\mathfrak{X}$ for the rank of $\mathfrak{L}_{\mathfrak{X}}$ in $\mathfrak{X}^{\mathfrak{X}}$ to be at most $\mathfrak{d}$. The complexity of the statement of the main theorem in this section reflects the variety of examples of spaces which satisfy its hypothesis. These examples include many of the most natural countable discrete metric spaces. For example, all subsets of $\mathbb{N}^k$ for all $k \in \mathbb{N}$ satisfy the hypothesis of the theorem.

Recall that if A, B are subsets of a metric space $\mathfrak{X}$ with metric d, then

$$\overline{d}(A, B) = \inf\{\, d(x, y) \;:\; x \in A, y \in B \,\}.$$

If $\mathcal{U}$ is a partition of $\mathfrak{X}$, then for $U, V \in \mathcal{U}$ define

$$d_{\mathcal{U}}(U, V) = \inf\{\, \overline{d}(V_1, V_2) + \cdots + \overline{d}(V_{i-1}, V_i) \;:\; V_j \in \mathcal{U}, V_1 = U, V_i = V, \text{ and } i \in \mathbb{N} \,\}.$$

It is straightforward to verify that $d_{\mathcal{U}}$ is a metric on $\mathcal{U}$. The main theorem of this section is the following.

Theorem 4.1. *Let $\mathfrak{X}$ be a countable discrete metric space with metric d. If there exists a partition $\mathcal{U} = \{U_1, U_2, \ldots\}$ of $\mathfrak{X}$ with $\inf_{i,j} d_{\mathcal{U}}(U_i, U_j) > 0$ and where every open ball in $(\mathcal{U}, d_{\mathcal{U}})$ is finite, then $\mathfrak{X}^{\mathfrak{X}}_{\leqslant} \preccurlyeq \mathfrak{L}_{\mathfrak{X}}$ and so* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \leqslant \mathfrak{d}$.

The following notion and lemma are used in the proof of Theorem 4.1. Let $x, y \in \mathfrak{X}$. Then any finite sequence $x = y_1, \ldots, y_n = y$ of not necessarily distinct points such that $d(y_i, y_{i+1}) \leqslant C$ for some fixed C and for all i is called a *C-chain from x to y.*

Lemma 4.2. *If there is no $C > 0$ such that every pair $x, y \in \mathfrak{X}$ can be connected by a C-chain, then $\mathfrak{X}^{\mathfrak{X}}_{\leqslant} \preccurlyeq \mathfrak{L}_{\mathfrak{X}}$ and so* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \leqslant \mathfrak{d}$.

Proof. Let $\mathfrak{X} = \{x_1, x_2, \ldots\}$ and let $i \in \mathbb{N}$. Define V_i to be the set of all points that can be reached from x_1 by an i-chain. By assumption, the sequence $V_i \setminus V_{i-1}$ contains infinitely many nonempty sets. Denote these nonempty sets by $U_1 = V_{m(1)} \setminus V_{m(1)-1}, U_2 = V_{m(2)} \setminus V_{m(2)-1}, \ldots$ for the appropriate (strictly increasing) sequence of naturals $\{m(j)\}_{j \in \mathbb{N}}$. Of course, $V_{m(k)} \setminus V_{m(k)-1} = V_{m(k)} \setminus V_{m(k-1)}$. Note that if $v \in U_j$, then there is a $m(j)$-chain from v to x_1 but not an i-chain for any $i < m(j)$. Fix $y_i \in U_i$ for each i.

Let $g \in \mathfrak{X}^{\mathfrak{X}}$ be the function defined by $g(x_i) = y_{n(i)}$ where $\{n(i)\}_{i\in\mathbb{N}}$ is any strictly increasing sequence satisfying

$$n(i) > \max\{\, d(x_j, x_k) \ : \ j, k \leqslant i \,\} + 1.$$

We will prove that $\langle\, \mathfrak{L}_{\mathfrak{X}}, g \,\rangle$ contains $\mathfrak{X}^{\mathfrak{X}}_{\leqslant}$.

Let $f \in \mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ be arbitrary. We will define a Lipschitz function λ such that $\lambda \circ g = f$. First, λ is defined on the set of all $y_{n(i)}$ by $\lambda(y_{n(i)}) = f(x_i)$. Although λ is not yet fully defined, $\lambda \circ g(x_i) = \lambda(y_{n(i)}) = f(x_i)$. The definition of λ is completed by defining $\lambda(y) = f(x_i)$ whenever $y \in U_j$ and $n(i) \leqslant j < n(i+1)$.

The proof is concluded by showing that λ is Lipschitz. Let $u, v \in \mathfrak{X}$ with $u \in U_i$, $v \in U_j$, $n(r) \leqslant i < n(r+1)$, and $n(s) \leqslant j < n(s+1)$. If $r = s$, then $\lambda(u) = f(x_r) = \lambda(v)$. If $r < s$, then

$$d(\lambda(u), \lambda(v)) = d(f(x_r), f(x_s)) \leqslant \max\{\, d(x_k, x_l) \ : \ k, l \leqslant s \,\} < n(s) - 1 \leqslant j - 1.$$

Now, there exists an $m(i)$-chain from x_1 to u and no $(m(j)-1)$-chain from x_1 to v. As $r < s$ it follows that $i < j$ and so $m(i) < m(j)$. Hence there is no $(m(j)-1)$-chain from u to v. In particular, $d(u, v) > m(j) - 1 > j - 1$ (the last inequality holds as $\{m(j)\}_{j\in\mathbb{N}}$ is strictly increasing). Hence λ is Lipschitz with constant 1. $\square$

If $U_1, U_2, \ldots$ are the sets used in the proof of Lemma 4.2, then $d_{\mathcal{U}}(U_i, U_j) > m(j) - 1$ whenever $j > i$. Hence $\inf_{i,j} d_{\mathcal{U}}(U_i, U_j) > m(2) - 1 \geqslant 1$. It follows that these sets satisfy the hypothesis of Theorem 4.1. Hence any space $\mathfrak{X}$ satisfying the hypothesis of Lemma 4.2 also satisfies that of Theorem 4.1.

Proof of Theorem 4.1. If $\mathfrak{X}$ satisfies the hypothesis of Lemma 4.2, then there is nothing to prove. Hence we may assume that there exists $C > 0$ such that for all $x, y \in \mathfrak{X}$ there is a C-chain from x to y.

The elements $x_1, x_2, \ldots$ of $\mathfrak{X}$ can be arranged into a sequence $y_1, y_2, \ldots$ with possible repetitions such that $d(y_i, y_{i+1}) < C$ for all i. Define $\phi : \mathfrak{X} \longrightarrow \mathbb{N}$ by

$$\phi(x_i) = \min\{\, j \ : \ x_i = y_j \,\}.$$

Let $n : \mathbb{N} \longrightarrow \mathbb{N}$ be any function satisfying

$$(1) \quad n(2i-1) - n(2i-2) > n(2i-2) - n(2i-3) > \max\{\phi(x_1), \phi(x_2), \ldots, \phi(x_i)\}.$$

For the sake of brevity, in what follows we will denote the balls $B_{\mathcal{U}}(U_1, Ck)$, $k \in \mathbb{N}$, of radius Ck around U_1 with respect to $d_{\mathcal{U}}$ by $B(k)$.

We will now prove that $B(k+1) \setminus B(k) \neq \emptyset$ for all $k \in \mathbb{N}$. Seeking a contradiction assume that there exists $k \in \mathbb{N}$ such that $B(k+1) \setminus B(k) = \emptyset$. Let $u \in U_1$, let $v \in U_i$ where $U_i \in \mathcal{U} \setminus B(k)$, and let $u = z_1, z_2, \ldots, z_n = v$ be any finite sequence in $\mathfrak{X}$. Then there exists i such that $z_i \in U_j$ with $U_j \in B(k)$ and $z_{i+1} \in U_r$ with $U_r \notin B(k)$. Since $B(k+1) \setminus B(k) = \emptyset$, it follows that $U_r \notin B(k+1)$. Hence

$$d(z_i, z_{i+1}) \geqslant \overline{d}(U_j, U_r) \geqslant d_{\mathcal{U}}(U_j, U_r) > C.$$

Thus there is no C-chain from u to v, a contradiction. Hence we infer that each of the sets $B(k+1) \setminus B(k)$ is nonempty. Thus we may fix $V_i \in B(n(2i-1)) \setminus B(n(2i-2))$ for all i and where $B(n(0)) = \emptyset$.

Let $v_i \in V_i$ be fixed for all i. Then define a function $g \in \mathfrak{X}^{\mathfrak{X}}$ by $g(x_i) = v_i$. We will eventually prove that $\mathfrak{X}^{\mathfrak{X}}_{\leqslant} \subseteq \langle\, \mathfrak{L}_{\mathfrak{X}}, g \,\rangle$.

To this end, let $f \in \mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ be arbitrary. Note that $f(x_i) = y_{\phi(f(x_i))}$ for all i. We will define a Lipschitz mapping λ whose composition with g is f. The definition is made in two stages. First, if $u \in U$ with $U \in B(n(2i-1)) \setminus B(n(2i-2))$, then define

$$\lambda(u) = y_{\phi(f(x_i))}. \tag{2}$$

In particular, $\lambda(v_i) = y_{\phi(f(x_i))} = f(x_i)$. Hence, although the definition of λ is not yet complete, we have that

$$\lambda(g(x_i)) = \lambda(v_i) = y_{\phi(f(x_i))} = f(x_i).$$

It only remains to complete the definition of λ and show that λ is Lipschitz.

The second stage of the definition of λ is made by defining λ on the points in $\mathfrak{X}$ not in any $U \in B(n(2i-1)) \setminus B(n(2i-2))$. Each such point v lies in some $V \in B(k+1) \setminus B(k) \subseteq B(n(2i)) \setminus B(n(2i-1))$. Let

$$r(k,i) = \frac{k - n(2i-1)}{n(2i) - n(2i-1)}[\phi(f(x_{i+1})) - \phi(f(x_i))] + \phi(f(x_i)).$$

Then define

$$\lambda(v) = y_{\lfloor r(k,i) \rfloor}. \tag{3}$$

We will now prove that λ is Lipschitz. We start by showing that for all k if $u \in U$ and $v \in V$ with $U \in B(k+1) \setminus B(k)$ and $V \in B(k+2) \setminus B(k+1)$, then $\lambda(u), \lambda(v) \in \{y_t, y_{t+1}\}$ for some t.

If $n(2i-2) < k+1 < n(2i-1)$, then $U, V \in B(n(2i-1)) \setminus B(n(2i-2))$. Thus $\lambda(u) = \lambda(v)$ by (2). If $k+1 = n(2i-1)$, then $U \in B(n(2i-1)) \setminus B(n(2i-2))$ and $V \in B(n(2i)) \setminus B(n(2i-1))$. Hence $\lambda(u) = y_{\phi(f(x_i))}$ and $\lambda(v) = y_{\lfloor r(k+1,i) \rfloor} = y_{\lfloor r(n(2i-1),i) \rfloor}$ by (2) and (3), respectively. But

$$r(n(2i-1),i) = \frac{n(2i-1) - n(2i-1)}{n(2i) - n(2i-1)}[\phi(f(x_{i+1})) - \phi(f(x_i))] + \phi(f(x_i)) = \phi(f(x_i)).$$

Therefore $\lambda(u) = \lambda(v)$.

If $k+1 = n(2i-2)$, then $U \in B(n(2i-2)) \setminus B(n(2i-3))$ and $V \in B(n(2i-1)) \setminus B(n(2i-2))$. Hence $\lambda(u) = y_{\lfloor r(k,i-1) \rfloor} = y_{\lfloor r(n(2i-2)-1,i-1) \rfloor}$ and $\lambda(v) = y_{\phi(f(x_i))}$ by (3) and (2), respectively. But

$$\begin{aligned} r(n(2i-2)-1, i-1) &= \frac{n(2i-2) - 1 - n(2i-3)}{n(2i-2) - n(2i-3)}[\phi(f(x_i)) - \phi(f(x_{i-1}))] + \phi(f(x_{i-1})) \\ &= \phi(f(x_i)) - \frac{\phi(f(x_i)) - \phi(f(x_{i-1}))}{n(2i-2) - n(2i-3)}. \end{aligned}$$

But since $f \in \mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ and by (1) we have that

$$\begin{aligned} |\phi(f(x_i)) - \phi(f(x_{i-1}))| &< \max\{\phi(f(x_i)), \phi(f(x_{i-1}))\} \\ &\leqslant \max\{\phi(x_1), \phi(x_2), \ldots, \phi(x_i)\} < n(2i-2) - n(2i-3). \end{aligned} \tag{4}$$

Therefore $\lfloor r(n(2i-2)-1, i-1) \rfloor = \phi(f(x_i)) - 1$ or $\phi(f(x_i))$, as required.

Finally, if $n(2i-1) < k+1 < n(2i)$, then $U, V \in B(n(2i)) \setminus B(n(2i-1))$. Hence $\lambda(u) = y_{\lfloor r(k,i) \rfloor}$ and $\lambda(v) = y_{\lfloor r(k+1,i) \rfloor}$ by (3). But

$$|r(k+1,i) - r(k,i)| = \frac{|\phi(f(x_{i+1})) - \phi(f(x_i))|}{n(2i) - n(2i-1)} < 1$$

by (4). Therefore $|\lfloor r(k+1,i) \rfloor - \lfloor r(k,i) \rfloor| \leqslant 1$ and so $\lambda(u)$ and $\lambda(v)$ are either the same or consecutive elements of the sequence $y_1, y_2, \ldots$.

Let $k \in \mathbb{N}$, $U \in B(k+1) \setminus B(k)$, and $V \in B(k+2) \setminus B(k+1)$ be arbitrary. We have shown that $\lambda(u), \lambda(v) \in \{y_t, y_{t+1}\}$ for all $u \in U$ and $v \in V$. Hence

$$d(\lambda(u), \lambda(v)) \leqslant C \leqslant \frac{C}{c} d(u, v)$$

where $c = \inf_{i,j} d_{\mathcal{U}}(U_i, U_j)$. On the other hand, if U is as before and $V \in B(l+1) \setminus B(l)$, $l > k+1$, then

$$d(u, v) \geqslant \overline{d}(U, V) \geqslant d_{\mathcal{U}}(U, V) \geqslant C(l-k-1) > C.$$

Thus

$$d(\lambda(u), \lambda(v)) \leqslant (l-k)C = (l-k-1)C + C \leqslant d(u, v) + C < 2d(u, v).$$

It follows that λ is Lipschitz with constant $\max\{2, C/c\}$. □

The following are straightforward corollaries of Theorems 3.1 and 4.1.

Corollary 4.3. *If every open ball in $\mathfrak{X}$ is finite and d is bounded below, then* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$.

Corollary 4.4. *If $\mathfrak{X}$ is any infinite subset of $\mathbb{N}^k$, for some $k \in \mathbb{N}$, with the usual euclidean metric, then* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$.

The next corollary demonstrates why elements of the partition $\mathcal{U}$ of $\mathfrak{X}$ were used in Theorem 4.1 rather than individual elements.

Corollary 4.5. *Let $\mathfrak{X}$ be a countably infinite subset of $\mathbb{R}$ that contains no Cauchy sequence. If the differences of infinitely many pairs of consecutive elements in $\mathfrak{X}$ are greater than some $c > 0$, then* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$.

Note that Corollary 4.5 applies to all sequences of partial sums of any divergent series of positive real numbers with infinitely many terms greater than some $c > 0$. Also note that the metrics of spaces satisfying the hypothesis of Corollary 4.5 are not necessarily bounded below.

The following lemma provides a further method of proving that $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$ for certain types of space $\mathfrak{X}$.

Theorem 4.6. *If every open ball in $\mathfrak{X}$ is finite and every Lipschitz mapping from any subset to itself can be extended to a Lipschitz mapping of the whole space, then $\mathfrak{L}_{\mathfrak{X}} \approx \mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ and so* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$.

Proof. That $\mathfrak{L}_{\mathfrak{X}} \preccurlyeq \mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ follows from Theorem 3.1.

To prove $\mathfrak{X}^{\mathfrak{X}}_{\leqslant} \preccurlyeq \mathfrak{L}_{\mathfrak{X}}$, note that since d is unbounded above on $\mathfrak{X}$ there exists a set $\mathcal{Y} = \{y_1, y_2, \ldots\}$ satisfying

$$d(y_{i+1}, y_1) \geqslant 2d(y_i, y_1)$$

for all i. Let $\mathfrak{X} = \{x_1, x_2, \ldots\}$. Then define $f \in \mathfrak{X}^{\mathfrak{X}}$ by $f(x_i) = y_i$ and let f^{-1} denote any extension of the inverse of f to an element of $\mathfrak{X}^{\mathfrak{X}}$.

Let $g \in \mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ be arbitrary, let $t : \mathbb{N} \longrightarrow \mathbb{N}$ be the function such that $g(x_i) = x_{t(i)}$, and let $h \in \mathcal{Y}^{\mathcal{Y}}$ be defined by $h(y_i) = y_{t(i)}$. Note that $t(i) \leqslant i$ for all i. Then for all $i < j$ we have that

$$\begin{aligned} d(h(y_i), h(y_j)) &\leqslant d(h(y_i), y_1) + d(h(y_j), y_1) \leqslant d(y_i, y_1) + d(y_j, y_1) \leqslant 2d(y_j, y_1) \\ &\leqslant 4d(y_j, y_1) - 4d(y_i, y_1) \leqslant 4d(y_j, y_i). \end{aligned}$$

Thus h is Lipschitz on $\mathcal{Y}$, and so, by assumption, can be extended to an element $\widehat{h}$ of $\mathfrak{L}_{\mathfrak{X}}$.

Now, for any $x_i \in \mathfrak{X}$

$$f^{-1}\widehat{h}f(x_i) = f^{-1}\widehat{h}(y_i) = f^{-1}(y_{t(i)}) = x_{t(i)} = g(x_i)$$

and $\mathfrak{X}^{\mathfrak{X}}_{\leqslant} \subseteq \langle\, \mathfrak{L}_{\mathfrak{X}}, f, f^{-1}\,\rangle$. □

By Corollary 4.4, we know that the rank of $\mathfrak{L}_{\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$, with the usual euclidean metric, is $\mathfrak{d}$. An alternative proof of this fact can be obtained using Theorem 4.6.

Corollary 4.7. *$\mathfrak{L}_{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}}_{\leqslant}$ and $\operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathfrak{L}_{\mathbb{N}}) = \mathfrak{d}$.*

Proof. The naturals $\mathbb{N}$ with the usual euclidean metric trivially satisfy the first condition of Lemma 4.6, that is, all open balls in $\mathbb{N}$ are finite. Thus it remains to prove that we can extend every Lipschitz mapping f from any subset A of $\mathbb{N}$ to A, to a Lipschitz mapping $\widehat{f}$ on the entire space. If the elements of A are $a_1 < a_2 < \cdots$, then the function $\widehat{f}$ defined by

$$\widehat{f}(x) = \begin{cases} \left\lfloor \frac{f(a_{i+1}) - f(a_i)}{a_{i+1} - a_i}(x - a_i) + f(a_i) \right\rfloor & a_i \leqslant x \leqslant a_{i+1} \\ f(a_1) & x \leqslant a_1 \end{cases}$$

is one possible such extension. □

Using the same argument as in the proof of Corollary 4.7, it is possible to prove that $\operatorname{rank}(\mathbb{Z}^{\mathbb{Z}} : \mathfrak{L}_{\mathbb{Z}}) = \mathfrak{d}$.

5. Countable Subsets of the Real Numbers

In this section we consider countably infinite discrete metric spaces arising as subsets of the real numbers $\mathbb{R}$. The following theorem is a straightforward consequence of the results in Sections 2 and 3.

Theorem 5.1. *Let $\mathfrak{X}$ be a countably infinite subset of the real numbers $\mathbb{R}$ with the usual euclidean metric. Then either*

(i) *$\mathfrak{X}$ contains a Cauchy sequence and $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = 1$; or*
(ii) *$\mathfrak{X}$ contains no Cauchy sequences and $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \geqslant \mathfrak{d}$.*

Proof. If $\mathfrak{X}$ contains a Cauchy sequence, then, by Theorem 2.5, $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = 1$. Otherwise every open ball in $\mathfrak{X}$ is finite and so, by Theorem 3.1, $\mathfrak{L}_{\mathfrak{X}} \preccurlyeq \mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ and $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \geqslant \mathfrak{d}$. □

However, in this section we will prove that $\mathfrak{L}_{\mathfrak{X}} \approx \mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ for certain types of subsets $\mathfrak{X}$ of $\mathbb{R}$. It will follow immediately that $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$.

Our original aim was to classify countable discrete subsets $\mathfrak{X}$ of $\mathbb{R}$ according to $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}})$. Although we have not completely accomplished this aim, we have succeeded in the sense that many natural examples of such spaces satisfy the condition of Theorem 5.4.

Every countably infinite subset $\mathfrak{X}$ of $\mathbb{R}$ that contains no Cauchy sequences can be given as $\cdots < z_{-1} < z_0 < z_1 < \cdots$ (not necessarily infinite in both directions). Recall that in previous sections we let $\mathfrak{X} = \{x_1, x_2, \ldots\}$. In this section we will use both of these enumerations of $\mathfrak{X}$ as appropriate, in particular, $\mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ continues to be defined in terms of the x_i.

Note that since $\mathfrak{X}$ is a subset of $\mathbb{R}$ with no Cauchy sequences, to prove that a mapping $f \in \mathfrak{X}^{\mathfrak{X}}$ is Lipschitz with constant $\lambda \in \mathbb{R}$ it suffices to prove that $|f(z_i) - f(z_{i+1})| \leqslant \lambda |z_i - z_{i+1}|$ for all i.

We will use the following lemma in the proof of the main theorem in this section.

Lemma 5.2. *If $\sigma = \{y_1, \ldots, y_M\} \subseteq \mathbb{R}$ and $y_1 < y_2 < \ldots < y_M$, then there exists a Lipschitz mapping $f : \sigma \longrightarrow \{y_1, y_M\}$ with constant at most M where $f(y_1) = y_1$ and $f(y_M) = y_M$.*

Proof. It is straightforward to see that there exists n with $1 \leqslant n < M$ and

$$|y_{n+1} - y_n| \geqslant \frac{1}{M}|y_M - y_1|.$$

Then

$$f(y_k) = \begin{cases} y_1 & 1 \leqslant k \leqslant n \\ y_M & n < k \leqslant M \end{cases}$$

is the desired function. □

Definition 5.3. Let $M, N \in \mathbb{N}$. Then an *(M,N)-expander of length k* is a set $\{z_{i(1)}, z_{i(2)}, \cdots, z_{i(k)}\}$ where $z_{i(1)} < z_{i(2)} < \cdots < z_{i(k)}$ and $|i(j+1) - i(j)| \leqslant M$ for all $1 \leqslant j \leqslant k-1$ and either

$$N|z_{i(m+1)} - z_{i(m)}| \geqslant |z_{i(n+1)} - z_{i(n)}| \text{ for all } 1 \leqslant m \leqslant n \leqslant k-1 \tag{5}$$

or

$$|z_{i(m+1)} - z_{i(m)}| \leqslant N|z_{i(n+1)} - z_{i(n)}| \text{ for all } 1 \leqslant m \leqslant n \leqslant k-1. \tag{6}$$

We will say that two expanders X and Y are *non-overlapping* if $\max(X) < \min(Y)$ or $\max(Y) < \min(X)$.

Theorem 5.4. *Let $\mathfrak{X}$ be a countably infinite subset of the real numbers $\mathbb{R}$ with the usual euclidean metric that contains no Cauchy sequences but does contain (M,N)-expanders of unbounded length for some fixed M and N. Then $\mathfrak{X}^{\mathfrak{X}}_{\leqslant} \approx \mathfrak{L}_{\mathfrak{X}}$ and so* $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$.

Proof. There are two steps in the proof. First, we will prove the theorem under the assumption that $\mathfrak{X}$ contains $(1,N)$-expanders of unbounded length satisfying (5). The proof of the theorem for the case of $(1,N)$-expanders of unbounded length satisfying (6) follows by an analogous argument.

Let $U_1, U_2, \ldots$ be $(1,N)$-expanders in $\mathfrak{X}$ and let

$$U_n = \{u^{(n)}_{-n}, u^{(n)}_{-n+1}, \ldots, u^{(n)}_{-1}, u^{(n)}_0, u^{(n)}_1, \ldots, u^{(n)}_n\}$$

where $u^{(n)}_{-n} < u^{(n)}_{-n+1} < \cdots < u^{(n)}_{-1} < u^{(n)}_0 < u^{(n)}_1 < \cdots < u^{(n)}_n$ for all n. Note that, as there are unbounded $(1,N)$-expanders in $\mathfrak{X}$, we may assume that $U_1, U_2, \ldots$ are non-overlapping.

Let $f \in \mathfrak{X}^{\mathfrak{X}}_{\leqslant}$ be arbitrary and let $t : \mathbb{N} \longrightarrow \mathbb{N}$ such that $f(x_i) = x_{t(i)}$. Note that $t(i) \leqslant i$ for all i from the definition of $\mathfrak{X}^{\mathfrak{X}}_{\leqslant}$. We define a function $g \in \mathfrak{X}^{\mathfrak{X}}$ by

$$g(x) = \begin{cases} u^{(n)}_{t(n)} & \text{if } x = u^{(n)}_i \text{ and } 0 \leqslant i \leqslant t(n) \\ u^{(n)}_{2i+t(n)} & \text{if } x = u^{(n)}_i \text{ and if } -t(n) < i < 0 \\ x & \text{otherwise.} \end{cases} \tag{7}$$

We will prove that g is an element of $\mathfrak{L}_{\mathfrak{X}}$. The mapping g fixes all the elements of $\mathfrak{X}$ outside $U_1, U_2, \ldots$ and all the points $u_i^{(n)}$ where $-n \leqslant i \leqslant -t(n)$ or $t(n) \leqslant i \leqslant n$. If $-t(n) \leqslant i < 0$, then

$$|g(u_{i+1}^{(n)}) - g(u_i^{(n)})| = |u_{2i+2+t(n)}^{(n)} - u_{2i+t(n)}^{(n)}| = |u_{2i+2+t(n)}^{(n)} - u_{2i+1+t(n)}^{(n)}|$$

$$+|u_{2i+1+t(n)}^{(n)} - u_{2i+t(n)}^{(n)}| \leqslant 2N|u_{2i+1+t(n)}^{(n)} - u_{2i+t(n)}^{(n)}| \leqslant 2N^2|u_{i+1}^{(n)} - u_i^{(n)}|.$$

If $0 \leqslant i < t(n)$, then $|g(u_i^{(n)}) - g(u_{i+1}^{(n)})| = 0$. Thus g is Lipschitz with constant $2N^2$ on the entire space $\mathfrak{X}$.

Let $h \in \mathfrak{X}^{\mathfrak{X}}$ be defined by $h(x_n) = u_0^{(n)}$ for all $n \in \mathbb{N}$ and let $k \in \mathfrak{X}^{\mathfrak{X}}$ be any mapping such that $k(u_i^{(n)}) = x_i$ for all $n \in \mathbb{N}$ and $0 \leqslant i \leqslant n$. Then

$$kgh(x_n) = kg(u_0^{(n)}) = k(u_{t(n)}^{(n)}) = x_{t(n)} = f(x_n).$$

Therefore $\mathfrak{X}^{\mathfrak{X}}_{\leqslant} \subseteq \langle\, \mathfrak{L}_{\mathfrak{X}}, h, k \,\rangle$ and so, by Theorem 3.1, $\mathfrak{X}^{\mathfrak{X}}_{\leqslant} \approx \mathfrak{L}_{\mathfrak{X}}$. It follows that $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$, as required.

The proof is concluded by showing that if $\mathfrak{X}$ contains (M,N)-expanders of unbounded length satisfying (5) and where $M > 1$, then $\mathfrak{X}^{\mathfrak{X}}_{\leqslant} \approx \mathfrak{L}_{\mathfrak{X}}$. As above let $U_1, U_2, \ldots$ be pairwise disjoint (M,N)-expanders in $\mathfrak{X}$ where

$$U_n = \{u_{-n}^{(n)}, u_{-n+1}^{(n)}, \ldots, u_{-1}^{(n)}, u_0^{(n)}, u_1^{(n)}, \ldots, u_n^{(n)}\}$$

for all n. In addition, let U_0 denote those points in $\mathfrak{X}$ that do not lie between elements in any U_i, that is $x \in U_0$ if whenever $y \geqslant x$ ($y \leqslant x$) for some $y \in U_i$ we have $z \geqslant x$ ($z \leqslant x$) for all $z \in U_i$.

Let $n, i \in \mathbb{N}$ and let $f_{n,i} : [u_i^{(n)}, u_{i+1}^{(n)}] \cap \mathfrak{X} \longrightarrow [u_i^{(n)}, u_{i+1}^{(n)}] \cap \mathfrak{X}$ be the Lipschitz function with constant M given by Lemma 5.2. Then the function $f : \mathfrak{X} \longrightarrow U_0 \cup U_1 \cup \cdots$ defined by

$$f(x) = \begin{cases} f_{n,i}(x) & x \in [u_i^{(n)}, u_{i+1}^{(n)}] \cap \mathfrak{X} \\ x & \text{otherwise} \end{cases}$$

is Lipschitz with constant M. Let $g : U_0 \cup U_1 \cup \cdots \longrightarrow U_0 \cup U_1 \cup \cdots$ be the function defined in (7). Then $g \circ f : \mathfrak{X} \longrightarrow \mathfrak{X}$ is a Lipschitz function with constant $2MN^2$ and by the same argument as that given above $\mathfrak{X}^{\mathfrak{X}}_{\leqslant} \subseteq \langle\, \mathfrak{L}_{\mathfrak{X}}, h, k \,\rangle$. □

In the previous section, using Theorem 3.1 we deduced that $\mathfrak{L}_{\mathbb{N}} \preccurlyeq \mathbb{N}^{\mathbb{N}}_{\leqslant}$ and so $\operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathfrak{L}_{\mathbb{N}}) \geqslant \mathfrak{d}$. In Corollary 4.7 we proved that the opposite inequalities also hold. Theorem 5.4 provides an alternative argument that $\mathbb{N}^{\mathbb{N}}_{\leqslant} \preccurlyeq \mathfrak{L}_{\mathbb{N}}$ since $\mathbb{N}$ contains unbounded $(1,1)$-expanders.

Example 5.5. Let $\mathfrak{H}$ be the sequence of partial sums of the harmonic series with the usual euclidean metric d_e, i.e. $\mathfrak{H} = \{\sum_{i=1}^{n} 1/i : n \in \mathbb{N}\}$. Since d_e is unbounded above on every infinite subset of $\mathfrak{H}$, it follows from Theorem 3.1 that $\mathfrak{L}_{\mathfrak{H}} \preccurlyeq \mathfrak{H}^{\mathfrak{H}}_{\leqslant}$ and $\operatorname{rank}(\mathfrak{H}^{\mathfrak{H}} : \mathfrak{L}_{\mathfrak{H}}) \geqslant \mathfrak{d}$. By Theorem 5.4, since $\mathfrak{H}$ is an infinite (1,1)-expander, it follows that the opposite inequalities also hold. That is, $\mathfrak{L}_{\mathfrak{H}} \approx \mathfrak{H}^{\mathfrak{H}}_{\leqslant}$ and $\operatorname{rank}(\mathfrak{H}^{\mathfrak{H}} : \mathfrak{L}_{\mathfrak{H}}) = \mathfrak{d}$.

6. Almost all subsets of $\mathbb{R}$

In the previous section, we showed that if $\mathfrak{X}$ is a countably infinite subset of $\mathbb{R}$ that does not contain a Cauchy sequence, then $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) \geqslant \mathfrak{d}$. In Sections 4 and 5, we established the equality

$$\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d} \tag{8}$$

for many of the commonly encountered subsets $\mathfrak{X}$ of $\mathbb{R}$. We have not succeeded in showing that (8) holds for all subsets $\mathfrak{X}$ of $\mathbb{R}$ with no Cauchy sequences. However, in this section we will show that (8) holds for almost all such subsets of $\mathbb{R} \setminus \{0\}$ in the sense of Baire category. The omission of 0 is by no means essential; we simply consider $\mathbb{R} \setminus \{0\}$ rather than $\mathbb{R}$ to avoid technicalities.

Let $\mathfrak{X}$ be a countably infinite subset of $\mathbb{R} \setminus \{0\}$ that does not contain a Cauchy sequence. Then $\mathfrak{X}$ is the disjoint union of a series of non-negative numbers and a series of non-positive numbers. (We follow the convention of defining a series as the sequence of its partial sums.) The first of these series corresponds to the elements of $\mathfrak{X} \cap (0, \infty)$ and the second corresponds to $\mathfrak{X} \cap (-\infty, 0)$. If $\mathfrak{X}$ has no negative or no positive elements, then the corresponding series is $\sum_{i=1}^{\infty} 0$. Obviously, instead of considering a sequence of non-positive terms we may consider, by taking its negative, two series of non-negative terms. As $\mathfrak{X}$ is infinite, the sequence of terms of one of these two series has only positive terms. We will think of $\mathfrak{X}$ as an element of the completely metrizable topological space $[0, \infty)^{\mathbb{N}} \times [0, \infty)^{\mathbb{N}}$ (with the usual product topology). Throughout the remainder of the section we will use $\mathbf{x}$ to denote a sequence $(x_i)_{i \in \mathbb{N}}$ in $[0, \infty)^{\mathbb{N}}$. We consider the following conditions on $\mathbf{x}$:

(i) $\sum_{i=1}^{\infty} x_i = \infty$ and $x_i > 0$ for all $i \in \mathbb{N}$;
(ii) there exists $N \in \mathbb{N}$ such that $x_i = 0$ for all $i \geqslant N$, and $x_i > 0$ for all $i < N$.

Let $\mathfrak{X}$ be a countably infinite subset of $\mathbb{R} \setminus \{0\}$ that does not contain a Cauchy sequence. Then in the next theorem we identify $\mathfrak{X}$, as described above, with an element $(\mathbf{x}, \mathbf{y}) \in [0, \infty)^{\mathbb{N}} \times [0, \infty)^{\mathbb{N}}$ such that either both $\mathbf{x}$ and $\mathbf{y}$ satisfy condition (i) or one satisfies condition (i) and the other condition (ii).

Theorem 6.1. *The family of all countably infinite subsets $\mathfrak{X}$ of $\mathbb{R} \setminus \{0\}$ with no Cauchy sequences satisfying $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$ is comeagre in $[0, \infty)^{\mathbb{N}} \times [0, \infty)^{\mathbb{N}}$.*

Proof. Let $\mathbb{A}$ denote the set of all $(\mathbf{x}, \mathbf{y}) \in [0, \infty)^{\mathbb{N}} \times [0, \infty)^{\mathbb{N}}$ where either both $\mathbf{x}$ and $\mathbf{y}$ satisfy condition (i) or one satisfies condition (i) and the other condition (ii). It suffices, by Corollary 4.5, to prove that the set

$$\mathbb{B} = \{\, (\mathbf{x}, \mathbf{y}) \in \mathbb{A} \;:\; y_i > 1 \text{ for infinitely many } i \,\}$$

is comeagre in $[0, \infty)^{\mathbb{N}} \times [0, \infty)^{\mathbb{N}}$.

We start by proving that the set

$$\mathbb{D} = \{\, \mathbf{x} \in [0, \infty)^{\mathbb{N}} \;:\; x_i > 1 \text{ for infinitely many } i \,\}$$

is dense G_δ in $[0, \infty)^{\mathbb{N}}$. Let $\mathbf{z} \in [0, \infty)^{\mathbb{N}}$ be arbitrary. Then the sequence $(\mathbf{z}^i)_{i \in \mathbb{N}}$ where $\mathbf{z}^i = (z_1, \ldots, z_i, 2, 2, \ldots) \in \mathbb{D}$ converges to $\mathbf{z}$. So, $\mathbb{D}$ is dense. If $j \in \mathbb{N}$ is arbitrary, then the set $\{\, \mathbf{z} \;:\; z_j \leqslant 1 \,\}$ is closed in $[0, \infty)^{\mathbb{N}}$. Hence

$$[0, \infty)^{\mathbb{N}} \setminus \mathbb{D} = \bigcup_{n=1}^{\infty} \bigcap_{j > n} \{\, \mathbf{z} \in [0, \infty)^{\mathbb{N}} \;:\; z_j \leqslant 1 \,\}$$

is F_σ and so $\mathbb{D}$ is G_δ.

In particular, it follows that the set $\mathbb{E} = \{\mathbf{x} \in [0,\infty)^{\mathbb{N}} : \mathbf{x}$ satisfies (i) or (ii)$\} \supseteq \mathbb{D}$ is comeagre in $[0,\infty)^{\mathbb{N}}$. Thus $\mathbb{E} \times \mathbb{D} = \mathbb{B}$ is comeagre in $[0,\infty)^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}}$. □

In the proof of Theorem 6.1 we used the fact that in almost all (in the sense of Baire category) sets considered the difference of infinitely many consecutive elements is greater than some fixed $c > 0$. We will now consider only those subsets of $\mathbb{R} \setminus \{0\}$ that do not have this property and show that the analogue of Theorem 6.1 holds in this restricted space.

Let $\mathfrak{X}$ be such a subset. Then, as above, $\mathfrak{X}$ is identified with $(\mathbf{x},\mathbf{y}) \in [0,\infty)^{\mathbb{N}} \times [0,\infty)^{\mathbb{N}}$ with the extra assumption that $\mathbf{x}$ and $\mathbf{y}$ tend to 0. If

$$c_0^+ = \{\, \mathbf{x} = (x_i)_{i\in\mathbb{N}} \in [0,\infty)^{\mathbb{N}} \;:\; (\forall i \in \mathbb{N})\ (x_i \geqslant 0) \text{ and } x_i \to 0 \,\},$$

then, in the following theorem, $\mathfrak{X}$ is identified with an element of $c_0^+ \times c_0^+$. Note that $c_0^+ \times c_0^+$ is completely metrizable with the metric

$$d(\mathbf{x},\mathbf{y}) = \max\{\, |x_i - y_i| \;:\; i \in \mathbb{N} \,\}$$

on c_0^+.

Theorem 6.2. *The family of all subsets $\mathfrak{X}$ of $\mathbb{R} \setminus \{0\}$ where $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$, and $\mathfrak{X}$ can be identified with an element of $c_0^+ \times c_0^+$ is comeagre in $c_0^+ \times c_0^+$.*

Proof. Consider the following sets:

$$\mathbb{A} = \{\, \mathbf{x} \in c_0^+ \;:\; (\forall i \in \mathbb{N})\ (x_i > 0) \,\}, \mathbb{B} = \{\, \mathbf{x} \in c_0^+ \;:\; \sum_{i=0}^{\infty} x_i = \infty \,\},$$
$$\mathbb{D} = \{\, \mathbf{x} \in c_0^+ \;:\; (\forall n)\ (\exists k)\ x_k > x_{k+1} > \cdots > x_{k+n} \,\}.$$

We will prove that $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{D}$ are all dense G_δ in c_0^+.

If $j \in \mathbb{N}$ is arbitrary, then the set $\mathbb{A}_j = \{\, \mathbf{x} \in c_0^+ \;:\; x_j > 0 \,\}$ is open in c_0^+. Let $\mathbf{x} \in c_0 \setminus \mathbb{A}_j$ be arbitrary and let $\epsilon > 0$. Then the sequence $\mathbf{y} \in \mathbb{A}_j$ with terms $y_i = x_i$, $i \neq j$, and $y_j = \epsilon/2$ satisfies $d(\mathbf{x},\mathbf{y}) = \epsilon/2 < \epsilon$. Hence $\mathbb{A}_j$ is dense in c_0^+. As $\bigcap_{j\in\mathbb{N}} \mathbb{A}_j = \mathbb{A}$, we have shown that $\mathbb{A}$ is dense G_δ in c_0^+.

Let $\mathbb{B}_{N,k} = \{\, \mathbf{x} \in c_0^+ \;:\; \sum_{i=1}^{k} x_i > N \,\}$, let $\mathbf{x} \in \mathbb{B}_{N,k}$, and let $\epsilon = (\sum_{i=1}^{k} x_i/k) - N/k$. Then if $\mathbf{y} \in c_0^+$ and $d(\mathbf{x},\mathbf{y}) < \epsilon$, then $\mathbf{y} \in \mathbb{B}_{N,k}$ and so $\mathbb{B}_{N,k}$ is open. It follows that $\mathbb{B}_N = \bigcup_{k=1}^{\infty} \mathbb{B}_{N,k}$ is also open.

Let $\mathbf{x} \in c_0^+$ and $\epsilon > 0$ be arbitrary. Then there exists $K \in \mathbb{N}$ such that $K > 1/\epsilon$. Let $\mathbf{y}$ be the sequence with terms $y_i = x_i + (1/K)$ if $i \leqslant NK$ and $y_i = x_i + (1/iK)$ if $i > NK$. Then $d(\mathbf{x},\mathbf{y}) = 1/K < \epsilon$ and $\sum_{i=1}^{NK} y_i = \sum_{i=1}^{NK} x_i + N > N$. Thus $\mathbf{y} \in \mathbb{B}_{N,NK}$ and so $\mathbb{B}_N$ is dense. So, $\mathbb{B}$ is dense G_δ, as $\bigcap_{N=1}^{\infty} \mathbb{B}_N = \mathbb{B}$.

Let $\mathbb{D}_n = \{\, \mathbf{x} \in c_0^+ \;:\; (\exists k)\ (x_k > x_{k+1} > \cdots > x_{k+n}) \,\}$. Let $\mathbf{x} \in \mathbb{D}_n$ be arbitrary, let $k \in \mathbb{N}$ such that $x_k > x_{k+1} > \cdots > x_{k+n}$, and let $\epsilon = \min\{\, x_{k+i+1} - x_{k+i} \;:\; 0 \leqslant i \leqslant n-1 \,\}$. Then $\mathbf{y} \in \mathbb{D}_n$ for all $\mathbf{y} \in c_0^+$ with $d(\mathbf{x},\mathbf{y}) < \epsilon/2$. Hence $\mathbb{D}_n$ is open and so $\mathbb{D} = \bigcap_{n=1}^{\infty} \mathbb{D}_n$ is G_δ in c_0^+.

It remains to prove that $\mathbb{D}$ is dense in c_0^+. Let $\delta > 0$ be arbitrary and let $\mathbf{x} \in c_0^+$. Then there exists m such that $x_i < \delta$ for all $i > m$. Let $\mathbf{y} \in c_0^+$ be defined by $y_i = x_i$ when $i \leqslant m$ and $y_i = \delta/i$ when $i > m$. It follows that $d(\mathbf{x},\mathbf{y}) < \delta$ and $\mathbf{y} \in \mathbb{D}$. Thus $\mathbb{D}$ is dense, as required.

It follows that $\mathbb{E} = \mathbb{A} \cap \mathbb{B} \cap \mathbb{D}$ is dense G_δ in c_0^+ and so $\mathbb{E} \times \mathbb{E}$ is dense G_δ in $c_0^+ \times c_0^+$. Note that every element of $\mathbb{D}$ (and hence $\mathbb{E}$) contains $(1,1)$-expanders of unbounded length. Hence if $\mathfrak{X} = (\mathbf{x}, \mathbf{y}) \in \mathbb{E} \times \mathbb{E}$, then $\mathfrak{X}$ satisfies $\operatorname{rank}(\mathfrak{X}^{\mathfrak{X}} : \mathfrak{L}_{\mathfrak{X}}) = \mathfrak{d}$ by Theorem 5.4. □

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