Residual Finiteness of Monoids, Associated Actions and Groups

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Residual Finiteness (1)

Definition

An algebraic structure $A$ is residually finite if any, and hence all, of the following equivalent conditions hold:

- For all $x, y \in A$, $x \neq y$, there exists a homomorphism $f: A \to B$, $B$ finite, such that $f(x) \neq f(y)$.

- For all $x, y \in A$, $x \neq y$, there exists a finite index congruence $\rho$ which separates $x$ and $y$, i.e. $(x, y) \not\in \rho$.

- The intersection of all finite index congruences of $A$ is trivial.

This applies to: groups, semigroups/monoids, actions, ...
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Examples
- finite structures (r.f. is a finiteness condition);
- free – semigroups/ groups/ commutative semigroups/ abelian groups/inverse semigroups;
- infinite simple (congruence free) structures are not r.f.

Facts
- Closed under taking substructures (obvious).
- Closed under finite index extensions (in groups), finite Rees index extensions (in semigroups, NR, Thomas), direct products (in general).
- A finitely presented, r.f. algebraic structure has a soluble word problem (Mostowski 66, Evans 70).
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Residual Finiteness (3)

- D. Segal, Residually finite groups, 1990.
- A series of papers by Golubov et al., 1970s.
Any monoid $S$ acts on itself via $(x, s) \mapsto xs$.

$\mathcal{R}$-classes := the strong orbits of this action.

$\mathcal{R}$ is a left congruence (i.e. $s \in S$ & $(x, y) \in \mathcal{R} \Rightarrow (sx, sy) \in \mathcal{R}$).

Hence, $S$ acts from the left on the set $S/\mathcal{R}$ of $\mathcal{R}$-classes.

Left/right duality $\rightarrow \mathcal{L}$-classes, right action on $S/\mathcal{L}$.

$\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. 

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Left/right duality $\Rightarrow \mathcal{L}$-classes, right action on $S/\mathcal{L}$.

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Example: $S, S/\mathcal{H}, S/\mathcal{R}, S/\mathcal{L}$

$$S = \langle a, b, c, h \mid \begin{array}{l}
aba = b, \ bab = a, \ c^3 = c, \ c^2 h = h, \ ch = ha^2, \\
ac = bc = ca = cb = ah = bh = hc = h^2 = 0 \rangle.$$
Schützenberger Groups (1)

Let $h \in S$, and let $H$ be the $H$-class of $h$.

$\text{Stab}_r(h) = \{ s \in S : Hs = H \} \leq S$.

$s \sim t \iff hs = ht$ – a congruence on $\text{Stab}_r(h)$.

$\Gamma_r(h) = \text{Stab}_r(h) / \sim$.

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Schützenberger Groups (2)

- $\Gamma_r(h)$ is a group acting regularly on $H$.
- $|\Gamma_r(h)| = |H|$.
- $\Gamma_r(h) \sim \Gamma_l(h')$.
- $(h, h') \in R \Rightarrow \Gamma_r(h) \sim \Gamma_l(h')$. 

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- \( \Gamma_r(h) \) is a group acting regularly on \( H \).
- \( |\Gamma_r(h)| = |H| \).
- \( \Gamma_r(h) \cong \Gamma_l(h) \).
- \( (h, h') \in \mathcal{R} \Rightarrow \Gamma_r(h) \cong \Gamma_l(h') \).
Idempotents, Subgroups, Regular Monoids

Fact
If \( h \in S \) is an idempotent then its \( H \)-class \( H \) is the largest subgroup of \( S \) containing \( h \) and \( \Gamma_r(h) \sim = H \).

Definition
A monoid \( S \) is regular if (\( \forall x \)) (\( \exists y \)) (\( xyx = x \)).

Fact
A monoid is regular iff every \( R \)-class contains an idempotent.
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A monoid is regular iff every \( \mathcal{R} \)-class contains an idempotent.
A monoid is 'composed' out of two actions (on $S/R$ and $S/L$) and a bunch of groups ($\Gamma_r(h), h \in H$).

Is the following in any sense true:

$S$ is residually finite if and only if the actions on $S/R$ and $S/L$, and all the Schützenberger groups $\Gamma_r(h)$ ($h \in H$) are residually finite?

After all: $S$ is finite iff $S/R$, $S/L$ and all $\Gamma_r(h)$ are finite and r.f. is a finiteness condition.
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Positive Results: Regular Monoids (1)

Definition

A monoid $S$ is of finite $J$-type if every $J$-class contains only finitely many $R$- and $L$-classes.

Theorem (Golubov 75)

Let $S$ be a regular monoid of finite $J$-type. Then $S$ is residually finite if and only if all its maximal subgroups (i.e. Schützenberger groups) are residually finite.

Proposition

If a regular monoid $S$ is of finite $J$-type then its action on its $R$-classes is also residually finite.
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Sketch of Proof
Let $L_s$, $L_t$ be two distinct $\mathcal{L}$-classes.
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The action of a residually finite regular monoid on its $L$-classes is residually finite.

Sketch of Proof
Let $L_s, L_t$ be two distinct $L$-classes.
Let $e, f$ be idempotents s.t. $e \in L_s$, $f \in L_t$. 
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Let $e, f$ be idempotents s.t. $e \in L_s, f \in L_t$.

General theory: $(e, f) \not\in \mathcal{L} \iff ef \neq e \lor fe \neq f$. 
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Wlog suppose \( ef \neq e \).
Separate \( e, ef \): \( \phi : S \rightarrow T, T \) finite, \( \phi(ef) \neq \phi(e) \).
Hence, in \( T, (\phi(e), \phi(f)) \notin \mathcal{L} \).
The action of \( S \) on the \( \mathcal{L} \)-classes of \( T \) is a finite homomorphic image of the action of \( S \) on \( S/\mathcal{L} \) which separates \( L_s \) and \( L_t \).
Positive Results: General Monoids

Theorem
If $S$ is a residually finite monoid then every Schützenberger group of $S$ is residually finite.

Sketch of Proof
(sketch) Let $s, t$ be distinct elements of $\Gamma_r(h)$. That means that $hs \neq ht$ in $S$.

Separate $hs, ht$: $\varphi: S \to T$, $T$ finite, $\varphi(hs) \neq \varphi(ht)$.

This induces $\Gamma_r(h) \to \Gamma_r(\varphi(h))$ which separates $s$ and $t$. 

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Nik Ruskuc: Residual Finiteness
Rees Matrix Semigroups (1)

Ingredients:
▶ a group $G$
▶ two index sets $I$, $J$
▶ $P = (p_{ji})_{j \in J, i \in I}$ – a $J \times I$ matrix with entries from $G$;

New semigroup: $S = M[G; I, J; P]$, on the set $I \times G \times J$, with multiplication $(i, g, j)(k, h, l) = (i, gp_{jk}h, l)$.
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Rees Matrix Semigroups (2)

\[ I \cong S/R \]

\[ J \cong S/L \]

Facts
▶ $S$ is regular.
▶ $L$ -classes are indexed by the set $J$.
▶ Every element acts as a constant mapping on $S/R$.
▶ The action of $S$ on $S/L$ is always residually finite.

Nik Ruskuc: Residual Finiteness
Rees Matrix Semigroups (2)

$J \cong S/L$

$J \cong S/L$

Facts

$S$ is regular.
Rees Matrix Semigroups (2)

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- $S$ is regular.
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Facts

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Rees Matrix Semigroups (2)

Facts

- $S$ is regular.
- $\mathcal{L}$-classes are indexed by the set $J$.
- Every element acts as a constant mapping on $S/\mathcal{L}$.
- The action of $S$ on $S/\mathcal{L}$ is always residually finite.
Theorem (Golubov 72) A Rees matrix semigroup $M = [G; I, J; P]$ is residually finite if and only if $G$ is residually finite and $P$ has only finitely many non-proportional rows and columns.

Corollary There exists a non-residually finite regular semigroup in which the actions on $S / R$ and $S / L$, and all the maximal subgroups are residually finite.
Rees Matrix Semigroups (3)

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Ingredients: A group $G$ and a normal subgroup $N \trianglelefteq G$. 
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Let $\overline{N} = \{ \overline{n} : n \in N \}$ be a copy of $N$.

New semigroup:

$$S(G, N) = \langle G, \overline{N}, h : \ hn = \overline{n}h, \ he_G = e_{\overline{N}}h = h, \ g\overline{n} = \overline{ng} = gh = h\overline{n} = 0 \ (g \in G, \ n \in N) \rangle.$$
Another Construction (2)

Facts

$S(G, N)$ is r.f. iff $G$ is r.f.

The action of $S(G, N)$ on its $L$-classes is r.f. iff $G/N$ is r.f.

Corollary

There exists a residually finite semigroup such that its action on the $L$-classes is not residually finite.

Nik Ruskuc: Residual Finiteness
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There exists a residually finite semigroup such that its action on the \( \mathcal{L} \)-classes is not residually finite.
One More Counter-example (1)

Let $M$ be the commutative monoid with presentation $\langle a, a^{-1}, b, c, d, e (i \in \mathbb{Z}) \mid a a^{-1} = a^{-1} a = 1, b_i c_j = d, b_i c_j = a^{\tau}(j-i), b_i b_j = b_i c_j = b_i d = b_i e = c_j c_k = c_j d = c_j e = dd = de = ee = 0 \rangle$.

The $R$-/$L$-classes of $M$ are:

- $A = \{a \pm p: p \in \mathbb{Z}\}$ – the group of units
- $B = \bigcup_{i \in \mathbb{Z}} B_i, B_i = Ab_i$
- $C = \bigcup_{i \in \mathbb{Z}} C_i, C_i = Ac_i$
- $D = Ad$, $E = Ae$, $\{0\}$. 

Nik Ruskuc: Residual Finiteness
One More Counter-example (1)

Let $M$ be the commutative monoid with presentation

$$\langle a, a^{-1}, b_i, c_i, d, e \ (i \in \mathbb{Z}) \mid \left. \begin{array}{l}
aa^{-1} = a^{-1}a = 1, \ b_ic_i = d, \ b_ic_j = a^{i-j}e \ (i \neq j), \\
bibj = b_ic_j = b_id = b_ie = c_je = dd = de = ee = 0 \\
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The action of $M$ on $M/L$ is not residually finite.

Nik Ruskuc: Residual Finiteness
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The action of $M$ on $M/\mathcal{L}$ is not residually finite.
We want $M$ to be residually finite.
One More Counter-example (3)

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We still have one free choice: the function $\tau : \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{Z}$.
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One choice that works is:

$$\tau(2^k(2r + 1)) = \frac{2}{3}(2^{2\lceil k/2 \rceil} - 1) \quad (k, r \in \mathbb{Z}, \ k \geq 0)$$
Let's introduce a really strong finiteness condition on the actions on $S/R$ and $S/L$:

That certainly guarantees that the two actions will be residually finite.

**Theorem**

Let $S$ be a monoid with finitely many left- and right ideals. Then $S$ is residually finite if and only if all its Schützenberger groups are residually finite.
Let's introduce a really strong finiteness condition on the actions on $S/R$ and $S/L$: finiteness itself.
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**Theorem**

*Let $S$ be a monoid with finitely many left- and right ideals. Then $S$ is residually finite if and only if all its Schützenberger groups are residually finite.*
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A Possible New Project

Definition

A an algebraic structure $A$ is residually free if for any $x, y \in A$ with $x \neq y$ there exists a homomorphism $f$ from $A$ into a free object $F$ such that $f(x) \neq f(y)$.
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Problem
Investigate residual freeness of semigroups and monoids. How does it depend on/affect residual freeness of its Schützenberger groups, and the actions on $R$- and $L$-classes?