Generating sets of Completely 0-Simple Semigroups

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**Rank**

**Definition** Let $S$ be a semigroup and let $T$ be a subset of $S$.

- The *rank* of $S$ is the smallest number of elements needed in order to generate $S$:

  $$\text{rank}(S) = \min\{|A| : \langle A \rangle = S\}.$$ 

- The *relative rank* of $S$ modulo $T$ is the minimal number of elements of $S$ that need to be added to $T$ in order to generate the whole of $S$:

  $$\text{rank}(S : T) = \min\{|A| : A \subseteq S, \langle T \cup A \rangle = S\}.$$
Example: the structure of $T_3$

\[
\begin{array}{ccc}
\{3\}, \{1, 2\} & \{1, 2\} & \{2, 3\} & \{1, 3\} \\
\{1\}, \{2, 3\} & [1, 2, 2], [2, 1, 1] & [2, 3, 3], [3, 2, 2] & [1, 3, 3], [3, 1, 1] \\
\{2\}, \{1, 3\} & [1, 2, 1], [2, 1, 2] & [2, 3, 2], [3, 2, 3] & [1, 3, 1], [3, 1, 3] \\
\{1, 2, 3\} & & & [1, 1, 1], [2, 2, 2], [3, 3, 3] \\
\end{array}
\]
Definition Let $J$ be some $\mathcal{J}$ class of a semigroup $S$. Then the principal factor of $S$ corresponding to $J$ is the set $J^* = J \cup \{0\}$ with multiplication

$$s \ast t = \begin{cases} st & : \text{if } s, t, st \in J \\ 0 & : \text{otherwise.} \end{cases}$$

Definition A semigroup with zero is called 0-	extit{simple} if $\{0\}$ and $S$ are its only ideals.

Theorem If $J$ is a $\mathcal{J}$ class of a semigroup $S$ then $J^*$ is either a 0-simple semigroup or else it is a zero semigroup.
Rees matrix semigroups

Definition

- \( G \) - a finite group.
Rees matrix semigroups

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- $I, \Lambda$ be non-empty finite index sets.
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- $P = (p_{\lambda_i})$ a regular $\Lambda \times I$ matrix over $G \cup \{0\}$. 
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- $G$ - a finite group.
- $I, \Lambda$ be non-empty finite index sets.
- $P = (p_{\lambda i})$ a regular $\Lambda \times I$ matrix over $G \cup \{0\}$.
- $S = (I \times G \times \Lambda) \cup \{0\}$ with multiplication

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} 
(i, gp_{\lambda j}h, \mu) : & p_{\lambda j} \neq 0 \\
0 : & \text{otherwise}
\end{cases}$$

$$(i, g, \lambda)0 = 0(i, g, \lambda) = 00 = 0.$$
Theorem (The Rees Theorem) A semigroup $S$ is completely 0-simple if and only if it is isomorphic to $\mathcal{M}^0[G; I, \Lambda; P]$ where $G$ is a group and $P$ is regular.
The BIG Problem

**Problem** Find a formula for the rank of an arbitrary completely 0-simple semigroup.

- What might we expect the value to depend on?
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- What might we expect the value to depend on?
- $|I|, |\Lambda|$.
- $\text{rank}(G)$. 
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- $E(S)$ (‘contribution’ from the entries in the matrix).
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- $E(S)$ (‘contribution’ from the entries in the matrix).
- Remember that $(i, p_{\lambda i}^{-1}, \lambda)$ are idempotent

\[
(i, p_{\lambda i}^{-1}, \lambda)(i, p_{\lambda i}^{-1}, \lambda) = (i, p_{\lambda i}^{-1}p_{\lambda i}p_{\lambda i}^{-1}, \lambda) = (i, p_{\lambda i}^{-1}, \lambda).
\]
Special Cases

We will break the problem up and consider the following special cases:

- Groups.
- Rectangular bands.
- Rectangular 0-bands - $\mathcal{M}^0[\{e\}; I, \Lambda; P]$.
- Simple semigroups.
- Connected 0-simple semigroups.
- Brandt semigroups ($P \sim I$).
Lemma Let $G$ be a finite group, then

$$\text{rank}(\mathcal{M}^0[G; \{1\}, \{1\}; (1)]) = \text{rank}(G).$$
Rectangular bands

Definition $R_{mn} = \{1, \ldots, m\} \times \{1, \ldots, n\}$ with

$$(i, j)(k, l) = (i, l).$$

Proposition

rank($R_{mn}$) = max{$m, n$}.

Proof
Rectangular 0-bands

Definition Let $I = \{1, 2, \ldots, m\}$ and $\Lambda = \{1, 2, \ldots, n\}$ be finite sets and let $P$ be a regular $n \times m$ matrix of 0s and 1s. A rectangular 0-band is a semigroup $S = ZB_{mn} = (I \times \lambda) \cup \{0\}$ whose multiplication is given by

$$(i, \lambda)(j, \mu) = \begin{cases} (i, \mu) & : \text{if } p_{\lambda j} = 1 \\ 0 & : \text{if } p_{\lambda j} = 0 \end{cases}$$

$$(i, \lambda)0 = 0(i, \lambda) = 00 = 0.$$
Rectangular 0-bands

Figure 1:

\[ P = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \]

\[ A = \{(1, 1), (2, 3), (3, 4), (4, 2)\} \]
Rectangular 0-bands

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(2, 3)
Rectangular 0-bands

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(2, 3)(1, 1)
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Theorem Let $S = ZB_{mn}$, be an $m \times n$ rectangular 0-band, then

$$\text{rank}(S) = \max\{m, n\}.$$
Corollary If $S = \mathcal{M}^0[G; I, \Lambda; P]$ is idempotent generated then

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\operatorname{rank}(S) = \max(|I|, |\Lambda|).
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Corollary If \( S = \mathcal{M}^0[G; I, \Lambda; P] \) is idempotent generated then
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\]

Corollary With
\[
K(n, r) = \{\alpha \in T_n : |\text{im}(\alpha)| \leq r\}, (2 \leq r \leq n - 1)
\]
we have
\[
\text{rank}(K(n, r)) = \max\left(\binom{n}{r}, S(n, r)\right) = S(n, r).
\]
Theorem (NR, 1994) Let $S = M[G; I, \Lambda; P]$ be a finite Rees matrix semigroup with $P$ in normal form. Then

$$\text{rank}(S) = \max(|I|, |\Lambda|, \text{rank}(G : H))$$

where $H = \langle P \rangle$.

Normal form

$$P = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & g_{22} & g_{23} & \ldots & g_{2n} \\
1 & g_{32} & g_{33} & \ldots & g_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & g_{n2} & g_{n3} & \ldots & g_{nn}
\end{pmatrix}$$
**Definition** Let $S = M^0[G; I, \Lambda; P]$, then we let $\Gamma(S)$ be the graph with set of vertices 
\[ \{(i, \lambda) \in I \times \Lambda : H_{i,\lambda} \text{ is a group}\} \] and $(i, \lambda)$ adjacent to $(j, \mu)$ if and only if $i = j$ or $\lambda = \mu$.

**Definition** We say $S = M^0[G; I, \Lambda; P]$ is connected if $\Gamma(S)$ is connected.

**Example** Connected.
Definition Let $S = M^0[G; I, \Lambda; P]$, then we let $\Gamma(S')$ be the graph with set of vertices
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Example Connected.
**Connected completely 0-simple semigroups**

**Definition** Let $S = \mathcal{M}^0[G; I, \Lambda; P]$, then we let $\Gamma(S')$ be the graph with set of vertices 
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**Example** Connected.
In particular \( S = \mathcal{M}[G; I, \Lambda; P] \) (simple semigroups) are all connected.
Connected completely 0-simple semigroups

**Definition** Let $S = \mathcal{M}^0[G; I, \Lambda; P]$, then we let $\Gamma(S')$ be the graph with set of vertices

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Example Disconnected.
Connected completely 0-simple semigroups

**Definition** Let \( S = \mathcal{M}^0[G; I, \Lambda; P] \), then we let \( \Gamma(S') \) be the graph with set of vertices \( \{(i, \lambda) \in I \times \Lambda : H_{i\lambda} \text{ is a group}\} \) and \((i, \lambda)\) adjacent to \((j, \mu)\) if and only if \( i = j \) or \( \lambda = \mu \).

**Definition** We say \( S = \mathcal{M}^0[G; I, \Lambda; P] \) is connected if \( \Gamma(S') \) is connected.

**Example**Disconnected.
Theorem (NR, 1994) Let $S = M^0[G; I, \Lambda; P]$ be a finite connected Rees matrix semigroup with regular matrix $P$ (in normal form). Then

$$\text{rank}(S) = \max(|I|, |\Lambda|, \text{rank}(G : H))$$

where $H$ is the subgroup of $G$ generated by the non-zero entries in $P$. 
Theorem (Howie, Gomes, 1986) Let 
\( B = B(G, \{1, \ldots, n\}) \) be a Brandt semigroup, where \( G \) is a finite group of rank \( r \). Then the rank of \( B \) (as an inverse semigroup) is \( r + n - 1 \).
**Theorem** (Howie, Gomes, 1986) Let $B = B(G, \{1, \ldots, n\})$ be a Brandt semigroup, where $G$ is a finite group of rank $r$. Then the rank of $B$ (as an inverse semigroup) is $r + n - 1$.

**Proof** ($\leq$)

$A = \{(1, g_1, 1), \ldots, (1, g_r, 1), (1, e, 2), (2, e, 3), \ldots, (n - 1, e, n)\}$

($\geq$) Using graph theory.
What do we know?

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**Brandt**

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Theorem Let $S = M^0[G; I, \Lambda; P]$ be a finite Rees matrix semigroup with regular matrix $P$ (in normal form) with connected components $C_1, \ldots, C_k$ and $H_j$ the subgroup of $G$ generated by all non-zero entries of $C_j$, for $j = 1, \ldots, k$. Then

$$\text{rank}(S) = \max(|I|, |\Lambda|, r_{\min} + k - 1)$$

where

$$r_{\min} = \min_{(g_1, \ldots, g_k) \in G \times \ldots \times G} \left( \text{rank}(G : \bigcup_{j=1}^k g_j^{-1} H_j g_j) \right) .$$