Connected-homogeneous graphs

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Homogeneous graphs

Definition
A graph $\Gamma$ is called homogeneous if any isomorphism between finite induced subgraphs extends to an automorphism of the graph.

Homogeneity is the strongest possible symmetry condition we can impose on a graph.

Example
The line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$ is a finite homogeneous graph.
Classification of finite homogeneous graphs

Gardiner classified the finite homogeneous graphs.

Theorem (Gardiner (1976))

A finite graph is homogeneous if and only if it is isomorphic to one of the following:

1. finitely many disjoint copies of a complete graph $K_r$ (or its complement, complete multipartite graph)
2. the pentagon $C_5$
3. line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$. 
An infinite homogeneous graph

**Definition (The random graph \( R \))**

Constructed by Rado in 1964. The vertex set is the natural numbers (including zero).

For \( i, j \in \mathbb{N}, i < j \), then \( i \) and \( j \) are joined if and only if the \( i \)th digit in \( j \) (in base 2, reading right-to-left) is 1.

**Example**

Since \( 88 = 8 + 16 + 64 = 2^3 + 2^4 + 2^6 \) the numbers less that 88 that are adjacent to 88 are just \( \{3, 4, 6\} \).

Of course, many numbers greater than 88 will also be adjacent to 88 (for example \( 2^{88} \)).
The random graph

Consider the following property of graphs:

(*) For any two finite disjoint sets $U$ and $V$ of vertices, there exists a vertex $w$ adjacent to every vertex in $U$ and to no vertex in $V$. 
The random graph

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**Theorem**

There exists a countably infinite graph $R$ satisfying property (*), and it is unique up to isomorphism. The graph $R$ is homogeneous.

**Existence.** The random graph $R$ defined above satisfies property (*).
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Uniqueness and homogeneity. Both follow from a back-and-forth argument. Property (*) is used to extend the domain (or range) of any isomorphism between finite substructures one vertex at a time.
Building homogeneous graphs: Fraïssé’s theorem

- The **age** of a graph $\Gamma$ is the class of isomorphism types of its finite induced subgraphs.

- e.g. the age of the random graph $R$ is the class of *all* finite graphs.
Building homogeneous graphs: Fraïssé’s theorem

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- e.g. the age of the random graph $R$ is the class of *all* finite graphs.

**Fraïssé (1953)** - gives necessary and sufficient conditions for a class $C$ of finite graphs to be the age of a countably infinite homogeneous graph $M$. The key condition is the **amalgamation property**.

If Fraïssé’s conditions hold, then $M$ is unique, $C$ is called a **Fraïssé class**, and $M$ is called the **Fraïssé limit** of the class $C$. 
Countable homogeneous graphs

Examples

- The class of all finite graphs is a Fraïssé class. Its Fraïssé limit is the random graph $R$.
- The class of all finite graphs not embedding $K_n$ (for some fixed $n$) is a Fraïssé class. We call the Fraïssé limit the countable generic $K_n$-free graph.

Theorem (Lachlan and Woodrow (1980))

Let $\Gamma$ be a countably infinite homogeneous graph. Then $\Gamma$ is isomorphic to one of: the random graph, a disjoint union of complete graphs (or its complement), the generic $K_n$-free graph (or its complement).
Connected-homogeneous graphs

**Definition**
A graph $\Gamma$ is **connected-homogeneous** if any isomorphism between connected finite induced subgraphs extends to an automorphism.

**Example**
The hexagon $C_6$ is connected-homogeneous

Use rotations and reflections
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**Example**

On the other hand the hexagon is not homogeneous.

There is no automorphism $\alpha$ such that $(u, v)^\alpha = (u, w)$. 
Connected-homogeneous graphs

Connected-homogeneity...

1. is a natural weakening of homogeneity;
2. gives a class of graphs that lie between the (already classified) homogeneous graphs and the (not yet classified) distance-transitive graphs.

\[ \text{homogeneous} \Rightarrow \text{connected-homogeneous} \Rightarrow \text{distance-transitive} \]

(A graph is **distance-transitive** if for any two pairs \((u, v)\) and \((u', v')\) with \(d(u, v) = d(u', v')\), where \(d\) denotes distance in the graph, there is an automorphism taking \(u\) to \(u'\) and \(v\) to \(v'\).)
Finite connected-homogeneous graphs

Gardiner classified the finite connected-homogeneous graphs.

Theorem (Gardiner (1978))

A finite graph is connected-homogeneous if and only if it is isomorphic to a disjoint union of copies of one of the following:

1. a finite homogeneous graph

2. bipartite “complement of a perfect matching”
   (obtained by removing a perfect matching from a complete bipartite graph $K_{s,s}$)

3. cycle $C_n$

4. the line graph $L(K_{s,s})$ of a complete bipartite graph $K_{s,s}$

5. Petersen’s graph

6. the graph obtained by identifying antipodal vertices of the 5-dimensional cube $Q_5$
Tree-like examples

Definition (Tree)
A **tree** is a connected graph without cycles. A tree is **regular** if all vertices have the same degree. We use $T_r$ to denote a regular tree of valency $r$.

**Fact.** A regular tree $T_r$ ($r \in \mathbb{N}$) is an example of an infinite locally-finite connected-homogeneous graph.

Definition (Semiregular tree)
$T_{a,b}$: A tree $T = X \cup Y$ where $X \cup Y$ is a bipartition, all vertices in $X$ have degree $a$, and all in $Y$ have degree $b$. 
Locally finite infinite connected-homogeneous graphs

Let \( r, l \in \mathbb{N} (l \geq 2) \)

Take the bipartite semiregular tree \( T_{r+1,l} \).

The graph \( X_{r,l} \) is given by:

**Vertices** = bipartite block of \( T_{r+1,l} \) of vertices of degree \( l \).

**Edges** = adjacent in \( X_{r,l} \) if their distance in the tree is 2.

(Macpherson (1982) proved that every connected infinite locally-finite distance transitive graph has this form)
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Infinite connected-homogeneous graphs

Theorem (RG, Macpherson (2007))

A countable graph is connected-homogeneous if and only if it is isomorphic to the disjoint union of a finite or countable number of copies of one of the following:

1. a finite connected-homogeneous graph;
2. a homogeneous graph;
3. the random bipartite graph;
4. bipartite infinite complement of a perfect matching;
5. the line graph of the infinite complete bipartite graph $K_{\aleph_0,\aleph_0}$;
6. a treelike graph $X_{\kappa_1,\kappa_2}$ with $\kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\}$. 
Future work

**Digraphs**

- There are $2^\aleph_0$ such graphs.

**Problem 1.** Classify the countably infinite connected-homogeneous digraphs.

**Problem 2.** Classify the locally-finite countably infinite connected-homogeneous digraphs.

**Recent progress (with R. Möller).**

In the case that the graph has more than one end we have:

1. a classification when the underlying graph embeds a triangle
2. underlying graph triangle-free $\Rightarrow$ digraph is highly-arc-transitive
   - can describe the descendants and the reachability graphs