Graphs with a high degree of symmetry

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Outline

Introduction
   Graphs, automorphisms, and vertex-transitivity

Two notions of symmetry
   Distance-transitive graphs
   Homogeneous graphs

An intermediate notion
   Connected-homogeneous graphs
Introduction

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Graphs and automorphisms

Definition

- A graph \( \Gamma \) is a pair \((V \Gamma, E \Gamma)\)
  - \( V \Gamma \) - vertex set
  - \( E \Gamma \) - set of 2-element subsets of \( V \Gamma \), the edge set.
- If \( \{u, v\} \in E \Gamma \) we say that \( u \) and \( v \) are adjacent writing \( u \sim v \).
- The neighbourhood of \( u \) is \( \Gamma(u) = \{v \in V \Gamma : v \sim u\} \), and the degree (or valency) of \( u \) is \(|\Gamma(u)|\).
- A graph \( \Gamma \) is finite if \( V \Gamma \) is finite, and is locally-finite if all of its vertices have finite degree.
- An automorphism of \( \Gamma \) is a bijection \( \alpha : V \Gamma \to V \Gamma \) sending edges to edges and non-edges to non-edges. We write \( G = \operatorname{Aut} \Gamma \) for the full automorphism group of \( \Gamma \).
Graphs with symmetry

Roughly speaking, the ‘more’ symmetry a graph has the ‘larger’ its automorphism group will be (and vice versa).

**Aim.** To obtain classifications of families of graphs with a high degree of symmetry.

In each case we impose a symmetry condition $\mathcal{P}$ and then attempt to describe all (countable) graphs with property $\mathcal{P}$.

For each class, this naturally divides into three cases:
- finite graphs;
- infinite locally-finite graphs;
- infinite non-locally-finite graphs.
Vertex-transitive graphs

Definition
Γ is vertex transitive if $G$ acts transitively on $V\Gamma$. That is, for all $u, v \in V\Gamma$ there is an automorphism $\alpha \in G$ such that $u^\alpha = v$.

This is the weakest possible condition and there are many examples.

Complete graph $K_r$ has $r$ vertices and every pair of vertices is joined by an edge.

Cycle $C_r$ has vertex set $\{1, \ldots, r\}$ and edge set $\{\{1, 2\}, \{2, 3\}, \ldots, \{r, 1\}\}$.

Empty graph $I_r$ is the complement of the complete graph $K_r$. (The complement $\Gamma$ of $\Gamma$ is defined by $V\Gamma = V\Gamma$, $E\Gamma = \{\{i, j\} : \{i, j\} \not\in E\Gamma\}$).
Some vertex transitive bipartite graphs

Definition
A graph is called bipartite if the vertex set may be partitioned into two disjoint sets $X$ and $Y$ such that no two vertices in $X$ are adjacent, and no two vertices of $Y$ are adjacent.

- **Complete bipartite** every vertex in $X$ is adjacent to every vertex of $Y$ (written $K_{a,b}$ if $|X| = a$, $|Y| = b$).

- **Perfect matching** there is a bijection $\pi : X \rightarrow Y$ such that $E_\Gamma = \{\{x, \pi(x)\} : x \in X\}$

- **Complement of perfect matching** $\{x, y\} \in E_\Gamma \iff y \neq \pi(x)$
Cayley graphs of groups

Definition
$G$ - group, $A \subseteq G$ a generating set for $G$ such that $1_G \not\in A$ and $A$ is closed under taking inverses (so $x \in A \Rightarrow x^{-1} \in A$).

The (right) Cayley graph $\Gamma = \Gamma(G, A)$ is given by

$$V\Gamma = G; \quad E\Gamma = \{\{g, h\} : g^{-1}h \in A\}.$$ 

Thus two vertices are adjacent if they differ in $G$ by right multiplication by a generator.

Fact. The Cayley graph of a group is always vertex transitive.
Example (Cayley graph of $S_3$)

$G =$ the symmetric group $S_3$,  \( A = \{ (1 2), (2 3), (1 3) \} \)

$\Gamma(G, A) \cong K_{3,3}$ a complete bipartite graph.
Vertex-transitive graphs

On the other hand, not every vertex transitive graph arises in this way.

Example (Petersen graph)
The Petersen graph is vertex transitive but is not a Cayley graph.

There are ‘far too many’ vertex transitive graphs for us to stand a chance of achieving a classification.
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Distance-transitive graphs

Definition
In a connected graph $\Gamma$ we define the distance $d(u, v)$ between $u$ and $v$ to be the length of a shortest path from $u$ to $v$.

Definition
A graph is distance-transitive if for any two pairs $(u, v)$ and $(u', v')$ with $d(u, v) = d(u', v')$, there is an automorphism taking $u$ to $u'$ and $v$ to $v'$.

distance-transitive $\Rightarrow$ vertex-transitive

Example
A connected finite distance-transitive graph of valency 2 is simply a cycle $C_n$. 
Definition
The **Hamming graph** $H(d, n)$. Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n - 1\}$. Then the vertex set of $H(d, n)$ is

$$
\mathbb{Z}_n^d = \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n
$$
d times

and two vertices $u$ and $v$ are adjacent if and only if they differ in exactly one coordinate.

The $d$-dimensional **hypercube** is defined to be $Q_d := H(d, 2)$. Its vertices are $d$-dimensional vectors over $\mathbb{Z}_2 = \{0, 1\}$.

**Fact.** $H(d, n)$ is distance transitive
Hypercubes $Q_i \ (i = 2, 3, 4)$
Finite distance-transitive graphs

The classification of the finite distance-transitive graphs is still incomplete, but a lot of progress has been made.

Definition
A graph is **imprimitive** if there is an equivalence relation on its vertex set which is preserved by all automorphisms.
Imprimitive distance-transitive graphs

The cube is imprimitive in two different ways.

1. **Bipartite** The bipartition relation

   \[ u \equiv v \iff d(u, v) \text{ is even} \]

   is preserved (2 equivalence classes: red and blue)

2. **Antipodal** The relation

   \[ u \approx v \iff u = v \text{ or } d(u, v) = 3 \]

   is preserved (4 equivalence classes: black, blue, purple and red)
Smith’s reduction

**Smith (1971)** showed that the *only* way in which a finite distance-transitive graph (of valency $> 2$) can be imprimitive is as a result of being bipartite or antipodal (as in the cube example above).

This reduces the classification of finite distance-transitive graphs to:

1. classify the finite primitive distance-transitive graphs (this is close to being complete, using the classification of finite simple groups; see recent survey by John van Bon in *European J. Combin.*);

2. find all ‘bipartite doubles’ and ‘antipodal covers’ of these graphs (still far from complete).
Infinite locally-finite distance-transitive graphs

Trees

**Definition (Tree)**
A **tree** is a connected graph without cycles. A tree is **regular** if all vertices have the same degree. We use $T_r$ to denote a regular tree of valency $r$.

**Fact.** A regular tree $T_r$ ($r \in \mathbb{N}$) is an example of an infinite locally-finite distance-transitive graph.

**Definition (Semiregular tree)**
$T_{a,b}$: A tree $T = X \cup Y$ where $X \cup Y$ is a bipartition, all vertices in $X$ have degree $a$, and all in $Y$ have degree $b$.

A semiregular tree will not in general be distance transitive.
Locally finite infinite distance-transitive graphs
A family of examples

- Let $r \geq 1$ and $l \geq 2$ be integers.
- Take a bipartite semiregular tree $T_{r+1,l}$
  - one block $A$ with vertices of degree $r + 1$
  - the other $B$ with vertices of degree $l$
- Define $X_{r,l}$
  - Vertex set = $B$
  - $b_1, b_2 \in B$ joined iff they are at distance 2 in $T_{r+1,l}$. 

$T_{3,4}$
Locally finite infinite distance-transitive graphs
A family of examples

Example $X_{r,l} = X_{2,4}$.

- Let $r = 2$ and $l = 4$.
- So $T_{r+1,l} = T_{3,4}$
  - $A = \text{vertices of degree 3 (in black)}$
  - $B = \text{vertices of degree 4 (in red)}$
- $X_{2,4}$
  - Vertex set $= B = \text{red vertices}$
  - $b_1, b_2 \in B$ joined iff they are at distance 2 in $T_{3,4}$.

$T_{3,4}$
Locally finite infinite distance-transitive graphs
A family of examples

Example $X_{r,l} = X_{2,4}$.

- Let $r = 2$ and $l = 4$.
- So $T_{r+1,l} = T_{3,4}$
  - $A =$ vertices of degree 3 (in black)
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- $X_{2,4}$
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- $X_{2,4}$
  - Vertex set $= B =$ red vertices
  - $b_1, b_2 \in B$ joined iff they are at distance 2 in $T_{3,4}$.
Macpherson’s theorem

The graphs $X_{\kappa_1,\kappa_2}$ ($\kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\}$) are distance transitive.

The neighbourhood of a vertex consists of $\kappa_2$ copies of the complete graph $K_{\kappa_1}$.

**Theorem (Macpherson (1982))**

A locally-finite infinite graph is distance transitive if and only if it is isomorphic to $X_{k,r}$ for some $k, r \in \mathbb{N}$.

The key steps in Macpherson’s proof are to take an infinite locally finite distance-transitive graph $\Gamma$ and

1. prove that $\Gamma$ is “tree-like” (i.e. show $\Gamma$ has infinitely many ends)
2. apply a powerful theorem of Dunwoody (1982) about graphs with more than one end
Non-locally-finite infinite distance-transitive graphs

On the other hand, for infinite non-locally-finite distance-transitive graphs far less is known.

The following result is due to Evans.

**Theorem**

There exist $2^{\aleph_0}$ non-isomorphic countable distance-transitive graphs.

**Proof.** Makes use of a construction originally due to Hrushovski (which is itself a powerful strengthening of Fraïssé’s method for constructing countable structures by amalgamation).
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Homogeneous graphs

Definition
A graph $\Gamma$ is called homogeneous if any isomorphism between finite induced subgraphs extends to an automorphism of the graph.

Homogeneity is the strongest possible symmetry condition we can impose.

$\text{homogeneous} \Rightarrow \text{distance-transitive} \Rightarrow \text{vertex-transitive}$
A finite homogeneous graph

Definition (Line graph)
The line graph $L(\Gamma)$ of a graph $\Gamma$ has vertex set the edge set of $\Gamma$, and two vertices $e_1$ and $e_2$ joined in $L(\Gamma)$ iff the edges $e_1, e_2$ share a common vertex in $\Gamma$.

Example
$L(K_{3,3})$ is a finite homogeneous graph
Classification of finite homogeneous graphs

Gardiner classified the finite homogeneous graphs.

**Theorem (Gardiner (1976))**

A finite graph is homogeneous if and only if it is isomorphic to one of the following:

1. finitely many disjoint copies of $K_r$ ($r \geq 1$) (or its complement);
2. The pentagon $C_5$;
3. Line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$. 
Infinite homogeneous graphs

Definition (The random graph $R$)
Constructed by Rado in 1964. The vertex set is the natural numbers
(including zero).

For $i, j \in \mathbb{N}$, $i < j$, then $i$ and $j$ are joined if and only if the $i$th digit in $j$
(in base 2) is 1.

Example
Since $88 = 8 + 16 + 64 = 2^3 + 2^4 + 2^6$ the numbers less that 88 that
are adjacent to 88 are just \{3, 4, 6\}. Of course, many numbers greater
than 88 will also be adjacent to 88 (for example $2^{88}$ will be).
The random graph

Consider the following property of graphs:

(*) For any two finite disjoint sets $U$ and $V$ of vertices, there exists a vertex $w$ adjacent to every vertex in $U$ and to no vertex in $V$.

**Theorem**

There exists a countably infinite graph $R$ satisfying property (*), and it is unique up to isomorphism. The graph $R$ is homogeneous.

**Existence.** The graph $R$ defined above satisfies property (*).

**Uniqueness and homogeneity.** Both follow from a back-and-forth argument. Property (*) is used to extend the domain (or range) of any isomorphism between finite substructures one vertex at a time.
Fraïssé’s theorem

Definition
A relational structure $M$ is homogeneous if any isomorphism between finite induced substructures of $M$ extends to an automorphism of $M$. The age of $M$ is the class of isomorphism types of its finite substructures.

Fraïssé (1953) showed how to recognise the existence of homogeneous structures from their ages.

A class $C$ is the age of a countable homogeneous structure $M$ if and only if $C$ is closed under isomorphism, closed under taking substructures, contains only countably many structures up to isomorphism, and satisfies the amalgamation property. If these conditions hold, then $M$ is unique, $C$ is called a Fraïssé class, and $M$ is called the Fraïssé limit of the class $C$. 
The amalgamation property says that two structures in $C$ with isomorphic substructures can be ‘glued together’, inside a larger structure of $C$, in such a way that the substructures are identified.

(AP) Given $B_1, B_2 \in C$ and isomorphism $f : A_1 \rightarrow A_2$ with $A_i \subseteq B_i$ ($i = 1, 2$), $\exists C \in C$ in which $B_1$ and $B_2$ are embedded so that $A_1$ and $A_2$ are identified according to $f$. 
Countable homogeneous graphs

Examples

- The class of all finite graphs is a Fraïssé class. Its Fraïssé limit is the random graph $R$.
- The class of all finite graphs not embedding $K_n$ (for some fixed $n$) is a Fraïssé class. We call the Fraïssé limit the countable generic $K_n$-free graph.

Theorem (Lachlan and Woodrow (1980))

Let $\Gamma$ be a countably infinite homogeneous graph. Then $\Gamma$ is isomorphic to one of: the random graph, a disjoint union of complete graphs (or its complement), the generic $K_n$-free graph (or its complement).
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Connected-homogeneous graphs

Distance-transitive graphs - classification incomplete

Homogeneous graph - classified

**Question.** Do there exist natural classes between homogeneous and distance-transitive that can be classified?

**Definition**
A graph $\Gamma$ is *connected-homogeneous* if any isomorphism between connected finite induced subgraphs extends to an automorphism.

$\text{homogeneous } \Rightarrow \text{connected-homogeneous } \Rightarrow \text{distance-transitive}$
Gardiner classified the finite connected-homogeneous graphs.

**Theorem (Gardiner (1978))**

A finite graph is connected-homogeneous if and only if it is isomorphic to a disjoint union of copies of one of the following:

1. a finite homogeneous graph
2. complement of a perfect matching
3. cycle $C_n$ ($n \geq 5$)
4. the line graph $L(K_{s,s})$ of a complete bipartite graph $K_{s,s}$ ($s \geq 3$)
5. Petersen’s graph
6. the graph obtained by identifying antipodal vertices of the 5-dimensional cube $Q_5$
Infinite connected-homogeneous graphs

Theorem (RG, Macpherson (in preparation))

Any countable connected-homogeneous graph is isomorphic to the disjoint union of a finite or countable number of copies of one of the following:

1. a finite connected-homogeneous graph;
2. a homogeneous graph;
3. the random bipartite graph;
4. the complement of a perfect matching;
5. the line graph of a complete bipartite graph \( K_{\mathbb{N}_0,\mathbb{N}_0} \);
6. a graph \( X_{\kappa_1,\kappa_2} \) with \( \kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\mathbb{N}_0\} \).

(The proof relies on the Lachlan-Woodrow classification of fully homogeneous graphs.)
Possible future work

Consider connected-homogeneity for other kinds of relational structure.

Schmerl (1979) classified the countable homogeneous posets. It turns out that weakening homogeneity to connected-homogeneity here essentially gives rise to no new examples.

Theorem (RG, Macpherson (in preparation))

A countable poset is connected-homogeneous if and only if it is isomorphic to a disjoint union of a countable number of isomorphic copies of some homogeneous countable poset.

The corresponding result for digraphs seems to be difficult.