Boundary index and finite presentability of subsemigroups

Robert Gray

University of St Andrews

International Conference on Semigroups and Languages
Lisbon 2005
Outline

1 Motivation
   - Subsemigroups: inheritance and extensions
   - The notion of index

2 Boundaries in Cayley graphs
   - Definitions and examples
   - Main results

3 Applications and concluding remarks
   - Applications
   - One sided boundaries and the converse
Subsemigroups and inheritance

Let $S$ be a semigroup with $T$ a subsemigroup of $S$.

Let $\mathcal{P}$ be a property of semigroups.

- $S$ satisfies $\mathcal{P} \Rightarrow T$ satisfies $\mathcal{P}$?
- $T$ satisfies $\mathcal{P} \Rightarrow S$ satisfies $\mathcal{P}$?

Some properties are passed from $S$ to all of its subsemigroups. (e.g. commutativity, finiteness, solvable word problem, ...)

Others are not. (e.g. being finitely generated / presented, automatic, having finite derivation type, ...)
Subsemigroups and inheritance

Let $S$ be a semigroup with $T$ a subsemigroup of $S$.

Let $\mathcal{P}$ be a property of semigroups.

- $S$ satisfies $\mathcal{P} \Rightarrow T$ satisfies $\mathcal{P}$?
- $T$ satisfies $\mathcal{P} \Rightarrow S$ satisfies $\mathcal{P}$?

Some properties are passed from $S$ to all of its subsemigroups. (e.g. commutativity, finiteness, solvable word problem, ...)

Others are not. (e.g. being finitely generated / presented, automatic, having finite derivation type, ...)
What is index?

Roughly speaking...

Index is a measure of the ‘size’ of $T$ inside $S$.

A ‘good’ definition of index should have the property that if $T$ is ‘big’ in $S$ then $S$ and $T$ share many properties.
Established notions of index

Subgroups of groups
Let $G$ be a group with $H$ a subgroup of $G$.

$[G : H] = \text{number of cosets of } H \text{ in } G$

Subsemigroups of semigroups
Let $S$ be a semigroup with $T$ a subsemigroup of $S$.

$[S : T]_R = |S \setminus T|$

We call this the Rees index of $T$ in $S$. 
Cayley Graphs

Definition

Let $S$ be a semigroup generated by a finite set $A$. The right Cayley graph $\Gamma_r(A, S)$ has:

- Vertices: elements of $S$.
- Edges: directed and labelled with letters from $A$.

Given an edge $e = s \xrightarrow{a} t \iff sa = t$

Given an edge $e = s \xrightarrow{a} t$ we define

$$\iota(e) = s, \quad \tau(e) = t$$

calling them the initial and terminal vertices of $e$. 
Semigroup boundaries

Definition

- The right boundary edges of $T$ in $S$:

$$\mathcal{E}_r(A, T) = \{ e \in \Gamma_r(A, S) : \iota(e) \notin T \text{ and } \tau(e) \in T \}$$
Semigroup boundaries

Definition

- **The right boundary edges** of $T$ in $S$:
  \[ \mathcal{E}_r(A, T) = \{ e \in \Gamma_r(A, S) : \iota(e) \not\in T \land \tau(e) \in T \} \]

- **The right boundary** of $T$ in $S$ is the set of terminal vertices of the right boundary edges, together with the generators of $A$ that belong to $T$. This set is given by:
  \[ \mathcal{B}_r(A, T) = U^1 A \cap T = \{ ua : u \in U^1, a \in A \} \cap T \]

- $S^1 = S$ with an identity adjoined
- $U = S \setminus T$ and $U^1 = S^1 \setminus T$
The picture to have in mind

\[ \Gamma_r(S) \]

- Blue edges: right boundary edges
- Red vertices: the right boundary of \( T \) in \( S \)
Semigroup boundaries

Definition

The **left boundary** is defined in the analogous way but using the left Cayley graph.

\[ B_l(A, T) = AU^1 \cap T = \{ au : u \in U^1, a \in A \} \cap T. \]

The **(two-sided) boundary** is the union of the left and right boundaries:

\[ B(A, T) = B_l(A, T) \cup B_r(A, T). \]
A (very) straightforward example

Example (Infinite cyclic group)

- $S = \mathbb{Z} = \langle -1, 1 \rangle$ (with operation $+$);
- $T = \mathbb{N} = \{0, 1, 2, \ldots \}$.

In this example we have:

$$B_l(\{-1, 1\}, T) = B_r(\{-1, 1\}, T) = B(\{-1, 1\}, T) = \{0, 1\}.$$
Another straightforward example

Example (Free monoid on two generators)

- \( S = \{a, b\}^* \), \( T = \{ \text{words that begin with the letter } a \} \).

Right boundary: \( B_r(\{a, b\}, T) = \{a\} \).
Another straightforward example

Example (Free monoid on two generators)

- \( S = \{a, b\}^* \), \( T = \{\text{words that begin with the letter } a\} \).

Left boundary: \( B_l(\{a, b\}, T) = \{a\} \cup \{ab\{a, b\}^*\} \).
Changing the generating set

Proposition

\[ S - \text{a finitely generated semigroup} \]

\[ T - \text{subsemigroup of } S \]

\[ A, B \subseteq S - \text{two finite generating sets for } S \]

\[ |B_r(A, T)| < \infty \iff |B_r(B, T)| < \infty \]

\[ |B_l(A, T)| < \infty \iff |B_l(B, T)| < \infty \]
Proposition

Let $S = \langle A \rangle$ where $|A| < \infty$ and let $T \leq S$. Then $T$ is generated by:

$$X = B_r(A, T) U^1 \cap T.$$ 

Moreover, the generating set $X$ is finite if $B(A, T)$ is finite.
Generating sets

Proposition

Let $S = \langle A \rangle$ where $|A| < \infty$ and let $T \leq S$. Then $T$ is generated by:

$$X = B_r(A, T) U^1 \cap T.$$  

Moreover, the generating set $X$ is finite if $B(A, T)$ is finite.

Theorem

If $S$ is finitely generated and $T$ has a finite boundary in $S$ then $T$ is finitely generated.
Example

- $S = \mathbb{Z} = \langle -1, 1 \rangle$ (with operation $+$);
- $T = \{5, 6, 7, \ldots \}$.

$B(A, T) = \{5\}$ and

$\langle B(A, T) \rangle = \{5, 10, 15, \ldots \} \neq T$
Generating set example

Example

- \( S = \mathbb{Z} = \langle -1, 1 \rangle \) (with operation \(+\));
- \( T = \{5, 6, 7, \ldots \} \).

\[
\langle X \rangle = \langle B_r(A, T)U^1 \cap T \rangle = \langle 5, 6, 7, 8, 9 \rangle = T
\]
Semigroup presentations

**Theorem**

*If $S$ is finitely presented and $T$ has a finite boundary in $S$ then $T$ is finitely presented.*
Semigroup presentations

**Theorem**

*If S is finitely presented and T has a finite boundary in S then T is finitely presented.*

**Corollary (Ruškuc 1998)**

*If S is finitely generated (resp. presented) and T has finite Rees index in S (i.e. $|S \setminus T| < \infty$) then T is finitely generated (resp. presented).*

**Proof.** *T has finite Rees index $\Rightarrow$ T has a finite boundary.*
Applications

Corollary (Folklore)

Let $S$ be a semigroup with $T$ a subsemigroup of $S$ such that $S \setminus T$ is an ideal. If $S$ is finitely generated (resp. presented) then $T$ is finitely generated (resp. presented).

**Proof.** $S \setminus T$ is an ideal $\Rightarrow$ $T$ has a finite boundary.

Corollary

Let $S = T \cup I$, a disjoint union, where $I$ is a two-sided ideal of $S$ and $T$ is a subsemigroup of $S$. If $S$ is finitely generated (resp. presented) and every orbit of $I$ under the action of $T$ is finite then $I$ is finitely generated (resp. presented).

**Proof.** Finite orbits $\Rightarrow$ $I$ has a finite boundary.
One sided boundaries and the converse

Theorem

Let $S$ be a finitely generated free semigroup and let $T$ be a finitely generated subsemigroup of $S$. If $T$ has a finite right boundary in $S$ then $T$ is finitely presented.

Theorem

Let $S = \bigcup_{i \in I} S_i$, a disjoint union, where $I$ is finite and each $S_i$ is a subsemigroup of $S$. If each $S_i$ is finitely presented and has a finite boundary in $S$ then $S$ itself is finitely presented.
For the future

- Find other interesting/natural examples of subsemigroups with finite boundaries.

- What other properties are inherited?
  - Being automatic
  - Having a finite complete rewriting system
  - Having finite derivation type