Generating Sets of Ideals of Endomorphism Monoids

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Setting the scene

Given a mathematical structure $M$ the set of endomorphisms of $M$ (written as $\text{End}(M)$) forms a monoid (i.e. a semigroup with identity).

Examples

- When $M = \{1, \ldots, n\}$ then $\text{End}(M) \cong T_n$ the full transformation semigroup.

- When $M = V$ an $n$-dimentional vector space then $\text{End}(M) \cong \text{GLS}(n, F)$ the general linear semigroup of all $n \times n$ matrices over the field $F$.

- When $M = Y_n$ an $n$-element chain then $\text{End}(M) \cong O_n$ the semigroup of order preserving mappings of $\{1, \ldots, n\}$. 
History

**Theorem** (Howie, 1966) Every singular map of $T_n$ is a product of idempotent maps (maps that satisfy $\alpha^2 = \alpha$):

$$\text{Sing}_n = \{ \alpha \in T_n : 1 \leq |\text{Im}(\alpha)| < n \}.$$ 

**Theorem** (Erdos, 1967) Every singular $n \times n$ matrix of GLS$(n, F)$ is a product of idempotent matrices (matrices $M$ satisfying $M^2 = M$):

$$\text{Sing}(V) = \{ A \in \text{End}(V) : 1 \leq \text{dim(Im}(A)) < n \}.$$ 

**Question** What is the smallest number of idempotent maps (matrices) that we need in order to generate all the singular maps (matrices)?
More generally

Given a finite idempotent generated semigroup \( S \):

1. What is the smallest number of elements required to generate \( S \)?

\[
\text{rank}(S) = \min\{|A| : \langle A \rangle = S\}.
\]

2. What is the smallest number of idempotents required to generate \( S \)?

\[
\text{idrank}(S) = \min\{|A| : A \subseteq E(S), \langle A \rangle = S\}.
\]

3. How do these numbers compare i.e. how much more difficult is it to generate \( S \) if we restrict our choice of generators to the set of idempotents?
A few more examples

Ideals of $T_n$ and $\text{End}(V)$

- $K(n, r) = \{ \alpha \in T_n : |\text{Im}(\alpha)| \leq r \}$;
- $I(r, n, q) = \{ A \in \text{End}(V) : \dim(\text{Im}(A)) \leq r \}$;

Order preserving maps

- $O_n = \{ \alpha \in \text{Sing}_n : (\forall x, y \in X_n) \quad x \leq y \Rightarrow x\alpha \leq y\alpha \}$;

Partial transformations

- $K'(n, r) = \{ \alpha \in P_n : |\text{Im}(\alpha)| \leq r \}$;
- $PO_n$ - partial order preserving transformations.
# Idempotent ranks

<table>
<thead>
<tr>
<th>Semigroup</th>
<th>Rank</th>
<th>Idrank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Sing}_n$</td>
<td>$\frac{n(n-1)}{2}$</td>
<td>$\frac{n(n-1)}{2}$</td>
</tr>
<tr>
<td>$\text{Sing}(V)$</td>
<td>$\frac{q^n-1}{q-1}$</td>
<td>$\frac{q^n-1}{q-1}$     where $q =</td>
</tr>
<tr>
<td>$K(n, r)$</td>
<td>$S(n, r)$</td>
<td>$S(n, r)$</td>
</tr>
<tr>
<td>$I(r, n, q)$</td>
<td>$\begin{bmatrix} n \ r \end{bmatrix}_q$</td>
<td>?</td>
</tr>
<tr>
<td>$K'(n, r)$</td>
<td>$S(n + 1, r + 1)$</td>
<td>$S(n + 1, r + 1)$</td>
</tr>
<tr>
<td>$O_n$</td>
<td>$n$</td>
<td>$2n - 2$</td>
</tr>
<tr>
<td>$PO_n$</td>
<td>$2n - 1$</td>
<td>$3n - 2$.</td>
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</tbody>
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Green’s relations

$S$ - semigroup, $x, y \in S$

$x \mathcal{R} y \iff xS^1 = yS^1$

$x \mathcal{L} y \iff S^1x = S^1y$

$x \mathcal{J} y \iff S^1xS^1 = S^1yS^1$

- $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} (= \mathcal{J})$
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
- $J_x \leq J_y \iff S^1xS^1 \subseteq S^1yS^1$
Green’s relations in $T_n$

$T_n$ - full transformation semigroup, \( \alpha, \beta \in T_n \)

- \( \alpha \mathcal{R} \beta \iff \text{Im}(\alpha) = \text{Im}(\beta) \)
- \( \alpha \mathcal{L} \beta \iff \ker(\alpha) = \ker(\beta) \)
- \( \alpha \mathcal{J} \beta \iff |\text{Im}(\alpha)| = |\text{Im}(\beta)| \)

\[
J_r = \{ \alpha \in T_n : |\text{Im}(\alpha)| = r \}
\]

\[
K(n, r) = \{ \alpha \in T_n : |\text{Im}(\alpha)| \leq r \} = J_r \cup \ldots \cup J_1.
\]

\[
K(n, r) = \langle J_r \rangle, \ 1 \leq r < n.
\]
Rees matrix semigroups

**Definition**

- $G$ - a finite group.
- $I, \Lambda$ - non-empty finite index sets.
- $P = (p_{\lambda i})$ a **regular** $\Lambda \times I$ matrix over $G \cup \{0\}$.
- $S = (I \times G \times \Lambda) \cup \{0\}$ with multiplication

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j} h, \mu) & : \ p_{\lambda j} \neq 0 \\ 0 & : \ \text{otherwise} \end{cases}$$

$$(i, g, \lambda)0 = 0(i, g, \lambda) = 00 = 0.$$
**Rectangular 0-bands**

**Definition** A rectangular 0-band is a 0-Rees matrix semigroup over the trivial group written as $\mathcal{M}^0[\{1\}; I, \Lambda; Q]$ where $Q$ is a regular $|\Lambda| \times |I|$ matrix over $\{0, 1\}$ and:

$$(i, \lambda)(j, \mu) = \begin{cases} (i, \mu) & : \quad p\lambda_j = 1 \\ 0 & : \quad \text{otherwise} \end{cases}$$

**Lemma** Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be an idempotent generated completely 0-simple semigroup and $T$ be the natural rectangular 0-band homomorphic image of $S$ ($q\lambda_i = 1 \iff p\lambda_i \neq 0$). Then

- $\text{rank}(S) = \text{rank}(T) = \max(|I|, |\Lambda|)$;
- $\text{idrank}(S) = \text{idrank}(T)$. 
Definition Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup. We let $\Delta(P)$ denote the undirected bipartite graph with set of vertices $I \cup \Lambda$ and an edge between $i$ and $\lambda$ if and only if $p_{\lambda i} \neq 0$.

$S = \mathcal{M}^0[G; I, \Lambda; P]$
There is a natural correspondence between non-zero products of idempotents in the rectangular 0-band $T$ and paths from $I$ to $\Lambda$ in $\Delta(Q)$.

Product : $(1, 2)(3, 3)(4, 1) = (1, 1)$ since $q_{23} = q_{34} = 1$. 

Path : 

Diagram of paths and products.
Question When does a square idempotent generated 0-Rees matrix semigroup $S$ satisfy:

$$\text{idrank}(S) = \text{rank}(S) = \max(|I|, |\Lambda|) = n?$$

**Necessary** $\Delta(P)$ has a perfect matching.

$\Lambda$ \quad $I$  
\[ \begin{array}{cccccc}  &  &  &  &  \\
* & * & * & * & * \\
\end{array} \]

Hall’s Condition: for all $X \subseteq I$, $|N(X)| \geq |X|$.

**Sufficient** $\Delta(P)$ is hamiltonian.

$\Lambda$ \quad $I$  
\[ \begin{array}{cccc}  &  &  &  \\
* & * & * & * \\
\end{array} \]

Every second edge of the hamiltonian circuit constitutes a generating set of idempotents with size $n$. 
Theorem Let $S = \mathcal{M}^{0}[G; I, \Lambda; P]$ be an idempotent generated completely 0-simple semigroup with $|I| = |\Lambda| = n$. Then the following are equivalent:

1. $\text{rank}(S) = \text{idrank}(S)$;

2. $S$ satisfies SHC. ($\emptyset \subsetneq X \subsetneq I, |\mathcal{N}(X)| > |X|$);

3. The minimum generating sets of $S$ are precisely the subsets that intersect every (non-zero) $\mathcal{R}$-class in exactly one place and every (non-zero) $\mathcal{L}$-class of $S$ in exactly one place.
Application to ideals of $\text{End}(V)$

Let $V$ be an $n$ dimensional vector space over the finite field $F$ where $|F| = q$. Let $J(r, n, q)$ denote the top $\mathcal{J}$-class of the ideal $I(r, n, q)$ where:

$$I(r, n, q) = \{ A \in \text{End}(V) : \dim(\text{Im}(A)) \leq r \};$$

Greens relations in $\text{End}(V)$ are given by:

$$ARB \iff \ker(A) = \ker(B);$$
$$ALB \iff \text{Im}(A) = \text{Im}(B);$$
$$ADB \iff \dim(\text{Im}(A)) = \dim(\text{Im}(B)).$$

Theorem (Dawlings, 1980)

$$\text{idrank}(I(n - 1, n, q)) = \text{rank}(I(n - 1, n, q)) = \frac{q^n - 1}{q - 1}.$$
\[ I(2, 4, 2) \subseteq GF(2)^4 \]
Uniform distribution of idempotents

**Definition** We will say that $S = \mathcal{M}^0[G; I, \Lambda; P]$ has a $k$-uniform distribution of idempotents if the graph $\Delta(P)$ is $k$-regular.

**Corollary** Every idempotent generated completely $0$-simple semigroup $S$ with $|I| = |\Lambda|$ and a $k$-uniform distribution of idempotents has an idempotent basis.

$$
\begin{align*}
\text{HC} & \iff \text{SHC} \iff \text{hamiltonian} \\
& \uparrow
\iff \text{UDI}
\end{align*}
$$
Corollaries

**Corollary** Let $V$ be an $n$ dimensional vector space over the finite field $F$ where $|F| = q$. Then:

$$\text{rank}(I(r, n, q)) = \text{idrank}(I(r, n, q)) = \begin{bmatrix} n \\ r \end{bmatrix}_q.$$

**Corollary** A subset of $I(r, n, q)$ is a generating set of minimum cardinality for $I(r, n, q)$ if and only if it consists of matrices of rank $r$ no two of which have the same nullspace or the same image space.
Independence Algebras

Definition An \textit{independence algebra} is an algebra (in the sense of universal algebra) that satisfies:

\[ \text{[E]} \text{ If } z \in \langle X \cup \{y\} \rangle \text{ and } z \notin \langle X \rangle \text{ then } y \in \langle X \cup \{z\} \rangle. \]

A minimal generating set is called a \textit{basis} for \( A \) and its size is the \textit{dimension} of \( A \). (they all have the same size by [E])

\[ \text{[F]} \text{ Any map from a basis of } A \text{ into } A \text{ can be extended to an endomorphism of } A. \]

Examples Both sets and vector spaces are examples of independence algebras. Chains satisfy [E] but not [F].

Definition \( K(n, r) = \{ \alpha \in \text{End}(A) : \dim(\text{Im}(\alpha)) \leq r \} \).

Theorem (Fountain and Lewin, 1990) If \( A \) is an independence algebra of finite rank \( n \), then

\[ K(n, r) = \langle E(J_r) \rangle \text{ for } r = 1, \ldots, n - 1. \]
**Independence Algebras**

**Theorem** Let $A$ be a finite independence algebra with dimension $n \geq 3$. Then:

$$idrank(K(n, r)) = \text{rank}(K(n, r)) \quad (r = 1, \ldots, n - 1).$$

The above result uses the classification of finite independence algebras given by Cameron and Szabó.

**Problem** Give a proof of the above result directly from the definition of independence algebra.