Connected-homogeneous graphs

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Homogeneous graphs

Definition
A graph $\Gamma$ is called homogeneous if any isomorphism between finite induced subgraphs extends to an automorphism of the graph.

Homogeneity is the strongest possible symmetry condition we can impose on a graph.

Example
The line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$ is a finite homogeneous graph.
Gardiner classified the finite homogeneous graphs.

**Theorem (Gardiner (1976))**

A finite graph is homogeneous if and only if it is isomorphic to one of the following:

1. finitely many disjoint copies of a complete graph $K_r$ (or its complement, complete multipartite graph)
2. the pentagon $C_5$
3. line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$. 
An infinite homogeneous graph

Definition (The random graph $R$)

Constructed by Rado in 1964. The vertex set is the natural numbers (including zero).

For $i, j \in \mathbb{N}$, $i < j$, then $i$ and $j$ are joined if and only if the $i$th digit in $j$ (in base 2, reading right-to-left) is 1.

Example

Since $88 = 8 + 16 + 64 = 2^3 + 2^4 + 2^6$ the numbers less that 88 that are adjacent to 88 are just \{3, 4, 6\}.

Of course, many numbers greater than 88 will also be adjacent to 88 (for example $2^{88}$).
The random graph

Consider the following property of graphs:

(*) For any two finite disjoint sets $U$ and $V$ of vertices, there exists a vertex $w$ adjacent to every vertex in $U$ and to no vertex in $V$. 
The random graph

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**Theorem**

There exists a countably infinite graph $R$ satisfying property (*), and it is unique up to isomorphism. The graph $R$ is homogeneous.

**Existence.** The random graph $R$ defined above satisfies property (*).
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**Existence.** The random graph $R$ defined above satisfies property (*).

**Uniqueness and homogeneity.** Both follow from a back-and-forth argument. Property (*) is used to extend the domain (or range) of any isomorphism between finite substructures one vertex at a time.
Building homogeneous graphs: Fraïssé’s theorem

- The **age** of a graph $\Gamma$ is the class of isomorphism types of its finite induced subgraphs.
- e.g. the age of the random graph $R$ is the class of *all* finite graphs.
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Fraïssé (1953) - gives necessary and sufficient conditions for a class $C$ of finite graphs to be the age of a countably infinite homogeneous graph $M$. The key condition is the amalgamation property.

If Fraïssé’s conditions hold, then $M$ is unique, $C$ is called a Fraïssé class, and $M$ is called the Fraïssé limit of the class $C$. 
Countable homogeneous graphs

Examples

- The class of all finite graphs is a Fraïssé class. Its Fraïssé limit is the random graph $R$.
- The class of all finite graphs not embedding $K_n$ (for some fixed $n$) is a Fraïssé class. We call the Fraïssé limit the countable generic $K_n$-free graph.

Theorem (Lachlan and Woodrow (1980))

Let $\Gamma$ be a countably infinite homogeneous graph. Then $\Gamma$ is isomorphic to one of: the random graph, a disjoint union of complete graphs (or its complement), the generic $K_n$-free graph (or its complement).
Connected-homogeneous graphs

Definition
A graph $\Gamma$ is connected-homogeneous if any isomorphism between connected finite induced subgraphs extends to an automorphism.

Connected-homogeneity...

1. is a natural weakening of homogeneity;
2. gives a class of graphs that lie between the (already classified) homogeneous graphs and the (not yet classified) distance-transitive graphs.

$\text{homogeneous} \Rightarrow \text{connected-homogeneous} \Rightarrow \text{distance-transitive}$
Gardiner classified the finite connected-homogeneous graphs.

**Theorem (Gardiner (1978))**

A finite graph is connected-homogeneous if and only if it is isomorphic to a disjoint union of copies of one of the following:

1. a finite homogeneous graph
2. bipartite "complement of a perfect matching" (the complement of the line graph $L(K_{2,n})$)
3. cycle $C_n$
4. the line graph $L(K_{s,s})$ of a complete bipartite graph $K_{s,s}$
5. Petersen’s graph
6. the graph obtained by identifying antipodal vertices of the 5-dimensional cube $Q_5$
Tree-like examples

Definition (Tree)
A tree is a connected graph without cycles. A tree is regular if all vertices have the same degree. We use $T_r$ to denote a regular tree of valency $r$.

Fact. A regular tree $T_r$ ($r \in \mathbb{N}$) is an example of an infinite locally-finite connected-homogeneous graph.

Definition (Semiregular tree)
$T_{a,b}$: A tree $T = X \cup Y$ where $X \cup Y$ is a bipartition, all vertices in $X$ have degree $a$, and all in $Y$ have degree $b$. 
Locally finite infinite connected-homogeneous graphs

Let \( r, l \in \mathbb{N} (l \geq 2) \)

Take the bipartite semiregular tree \( T_{r+1,l} \).

The graph \( X_{r,l} \) is given by:

**Vertices** = bipartite block of \( T_{r+1,l} \) of vertices of degree \( l \).

**Edges** = adjacent in \( X_{r,l} \) if their distance in the tree is 2.

(Macpherson (1982) proved that every connected infinite locally-finite distance transitive graph has this form)
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Infinite connected-homogeneous graphs

Theorem (RG, Macpherson (2007))

A countable graph is connected-homogeneous if and only if it is isomorphic to the disjoint union of a finite or countable number of copies of one of the following:

1. a finite connected-homogeneous graph;
2. a homogeneous graph;
3. the random bipartite graph;
4. bipartite infinite complement of a perfect matching;
5. the line graph of the infinite complete bipartite graph $K_{\aleph_0, \aleph_0}$;
6. a treelike graph $X_{\kappa_1, \kappa_2}$ with $\kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\}$. 
Possible future work

Consider connected-homogeneity for other kinds of relational structure.

**Posets**

Schmerl (1979) classified the countable homogeneous posets.

**Theorem (RG, Macpherson (2007))**

A countable poset is connected-homogeneous if and only if it is isomorphic to a disjoint union of a countable number of isomorphic copies of some homogeneous countable poset.

**Digraphs**

**Open problem.** Classify the countably infinite connected-homogeneous digraphs.