Approaching cosets using Green’s relations and Schützenberger groups

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General question

How are the properties of a semigroup related to those of its substructures?

**Index in group theory**

$G$ - group, $H$ - subgroup of $G$

- The right cosets of $H$ in $G$ are the sets $\{Hg : g \in G\}$.
- The index $[G : H]$ is the number of (right) cosets of $H$ in $G$.

$G$ is the disjoint union of the cosets of $H$

Subgroups of finite index share many properties with their parent groups.
Finiteness conditions

Proposition
For groups the following finiteness conditions are preserved when taking finite index subgroups and when taking finite index extensions:

- finitely generated
- periodic
- residually finite
- finitely presented
- $FP_n$
- automatic
- locally finite
- soluble word problem
- finite derivation type.

Idea

- Develop a theory of index for semigroups
- Use it to gain a better understanding of the relationship between the properties of a semigroup and those of its substructures.
Subgroups of monoids: translational index

$S$ - monoid, $K$ - subgroup of $S$

- The right cosets of $K$ are the elements of the strong orbit of $K$ under the action of $S$ by right multiplication. So

$$Ks \ (s \in S) \text{ is a right coset } \iff \exists t \in S : Kst = K.$$  

- The (right) translational index of $K$ is the number of right cosets.

Green’s relations interpretation

$H$ - the $\mathcal{H}$-class of $S$ containing $K$, $R$ - the $\mathcal{R}$-class containing $K$.

- The right cosets of $K$ partition $R$
- $K$ has finite translational index iff $[H : K] < \infty$ and $R$ contains only finitely many $\mathcal{H}$-classes.
Subgroups of monoids: translational index

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Green’s relations interpretation

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Properties inherited

$S$ - monoid, $K$ - subgroup of $S$ with finite translational index

**Theorem (Ruskuc (1999))**

If $S$ is finitely presented (resp. generated) then $K$ is finitely presented (resp. generated).

- Steinberg (2003) - for inverse semigroups gives an alternative proof using a topological approach.

**Higher dimensions - Finite derivation type**

- Is a property of finitely presented semigroups
- Can be thought of as a higher dimensional version of the property of being finitely presented. Think “relations between relations”
- Originated from work of C. Squier on finite complete rewriting systems

**Theorem (RG & Malheiro (2007))**

If $S$ has finite derivation type then $K$ has finite derivation type.
The converse

Cosets are local

- In general the properties of a single subgroup $K$ of finite index in $S$ will not influence the properties of $S$.
- e.g. Adjoin an extra identity 1 to $S$, let $K = \{1\}$, and consider $K \leq S^1$.

Covering the semigroup

$S$ - monoid, and suppose there exists a finite collection $K_1, \ldots, K_r$ of subgroups:

- each $K_i$ has finite translational index
- the cosets of $K_1 \ldots, K_r$ cover $S$.

For a given property $\mathcal{P}$ we can ask:

“If all $K_i$ have property $\mathcal{P}$ does it follow that $S$ has property $\mathcal{P}$?”
Regular semigroups

Let $S$ be a regular semigroup with finitely many left and right ideals.

**Theorem (Ruskuc (1999))**

*S is finitely presented (resp. finitely generated) if and only if all its maximal subgroups are finitely presented (resp. finitely generated).*

**Theorem (Golubov (1975))**

*S is residually finite if and only if all its maximal subgroups are residually finite.*

- Actually, Golubov proved a stronger result, assuming only that each $J$-class contains finitely many $L$- and $R$-classes.

**Question** What about arbitrary (non-regular) semigroups?
Schützenberger groups
...the groups that never were

Given an $\mathcal{H}$-class $H$ of any semigroup $S$ we can associate a group with $H$.

- Let $S$ be a monoid and let $H$ be any $\mathcal{H}$-class of $S$.
- $T(H) = \{ s \in S : Hs = H \}$: the stabilizer of $H$ in $S$.
- The relation $\sim$ on $T(H)$ defined by:
  \[ x \sim y \Leftrightarrow (\forall h \in H)(hx = hy) \]

  is a congruence.
- $\Gamma(H) = T(H)/\sim$ is a group, called the Schützenberger group of $H$.

**Basic properties**

- $|\Gamma(H)| = |H|$.
- If $H_1$ and $H_2$ belong to the same $\mathcal{D}$-class then $\Gamma(H_1) \cong \Gamma(H_2)$.
- If $H$ is a group then $\Gamma(H) \cong H$. 
Non-regular semigroups

Let $S$ be a monoid with finitely many left and right ideals.

**Theorem (Ruskuc (2000))**

*S is finitely presented (resp. finitely generated) if and only if all its Schützenberger groups are finitely presented (resp. finitely generated).*

**Theorem (RG & Ruskuc 2007)**

*S is residually finite if and only if all its Schützenberger groups are residually finite.*

- Applies to non-regular semigroups :-) 
- But does not generalise Golubov :-(

Wallace (1962) - developed a theory of relative Green’s relations

$S$ - semigroup, $T$ - subsemigroup of $S$, $a, b \in S$.

**Definition (Relative Green’s relations)**

\[
\begin{align*}
    aL^T b & \iff T^1 a = T^1 b, \\
    aR^T b & \iff aT^1 = bT^1, \\
    \mathcal{H}^T & = \mathcal{R}^T \cap \mathcal{L}^T.
\end{align*}
\]

- The $\mathcal{L}^T$-classes are the strong orbits under the action of $T^1$ on $S$ by left multiplication (dually for $\mathcal{R}^T$).
- $\mathcal{L}^T$, $\mathcal{R}^T$, and $\mathcal{H}^T$ are equivalence relations.
- $T$ is a union of $\mathcal{R}^T$-classes and is a union of $\mathcal{L}^T$-classes.

**Connection with cosets in group theory**

$G$ - group, $H$ - subgroup.

- $\{ \mathcal{R}^H \text{-classes} \} = \{ \text{left cosets of } H \}$
- $\{ \mathcal{L}^H \text{-classes} \} = \{ \text{right cosets of } H \}$
Green index

$S$ - semigroup, $T$ subsemigroup of $S$

**Definition**
The Green index of $T$ in $S$ is the number of $\mathcal{H}^T$-classes of $S \setminus T$.

**Example (Finite Green index)**

**Definition** The Rees index of $T$ in $S$ is $|S \setminus T|$

- $T$ has finite Rees index in $S \Rightarrow T$ has finite Green index in $S$.

$G$ - group, $H$ - subgroup of $G$

- $H$ has finite Green index $\Leftrightarrow H$ has finite group-theoretic index.

**Generalised Schützenberger groups**

Associated with each $\mathcal{H}^T$-class $H_i$ is a group $\Gamma_i$ arising from the action of $T$ on $H_i$.

**Question** How are the properties of $S$ related to those of $T$ and the groups $\Gamma_i$?
Green index

$S$ - semigroup, $T$ subsemigroup of $S$

**Definition**

The **Green index** of $T$ in $S$ is the number of $H^T$-classes of $S \setminus T$.

**Example (Finite Green index)**

**Definition** The **Rees index** of $T$ in $S$ is $|S \setminus T|$

- $T$ has finite Rees index in $S \Rightarrow T$ has finite Green index in $S$.

$G$ - group, $H$ - subgroup of $G$

- $H$ has finite Green index $\Leftrightarrow H$ has finite group-theoretic index.

- If $T$ has finite Rees index in $S$ then all the groups $\Gamma_i$ are finite.
- If $S = G$ and $T = N \trianglelefteq G$ then $\Gamma_i \cong N$ for all $i \in I$.

In both cases $T$ has $\mathcal{P} \Rightarrow \Gamma_i$ has $\mathcal{P}$ for all $i$. ($\mathcal{P}$ = any finiteness condition)
Green index results

$S$ - semigroup, $T$ - subsemigroup with finite Green index

$H_i \ (i \in I)$ - relative $\mathcal{H}$-classes in $S \setminus T$

$\Gamma_i \ (i \in I)$ - generalised Schützenberger groups

**Theorem (RG and Ruskuc (2006))**

$S$ is finitely generated (resp. periodic, locally finite) if and only if $T$ is finitely generated (resp. periodic, locally finite), in which case all the groups $\Gamma_i$ are finitely generated (resp. periodic, locally finite).

**Theorem (RG & Ruskuc (2006))**

$S$ is residually finite if and only if $T$ and all the groups $\Gamma_i$ are residually finite.

- There is an example of a semigroup $S$, with a finite Green index subsemigroup $T \leq S$ such that $T$ is residually finite but $S$ is not.
Green index results

$S$ - semigroup, $T$ - subsemigroup with finite Green index.

**Theorem (RG & Ruskuc (2006))**

*If $T$ is finitely presented and each group $\Gamma_i$ is finitely presented then $S$ is finitely presented.*

- These results provide common generalisations of the corresponding results for finite index subgroups of groups, and finite Rees index subsemigroups of semigroups.
- e.g. for finite generation we obtain a common generalisation of Schreier’s lemma for groups and Jura (1978) for semigroups.

**Open problem.** Prove that if $S$ is finitely presented then $T$ is finitely presented and each group $\Gamma_i$ is finitely presented.